Lecture 6

Controllability/observability, multivariable poles/zeros

6.1 Introduction

During the first five lectures we have reconsidered several of the mathematical tools and methods from the introductory control course and also extended a few of them to the context of multi-variable systems. We are now shifting our focus to design of multivariable control systems. For such systems it is harder to illustrate all relevant specifications in one graph and to keep track of all relevant variables. Hence there is a stronger need for systematic methods to organize information and computations efficiently.

Before introducing optimization methods for controller design, we will study how the achievable control performance is limited by properties of the plant. For this purpose, we will introduce concepts to measure the degree of *controllability* and *observability* of a plant and we will define *multivariable poles and zeros*. Finally we study minimal state-space realizations of a given transfer matrix.

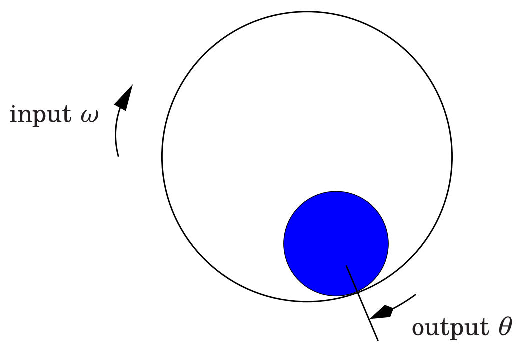


Figure 6.1 Example: Ball in the Hoop

Example 1 Consider a system where a ball is rolling inside a hoop as in Figure 6.1. The dynamics of the ball can be described by the equation

$$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$$

Two questions will be asked about this system:

1. Starting with the ball at rest in the bottom of the hoop ($\dot{\theta}(0) = \theta(0) = 0$), is it possible to bring the ball to $\theta(T) = \pi/4$ with $\dot{\theta}(T) = 0$ using the rotational speed ω of the hoop as input?
2. Using the same input, is it possible to keep the ball at $\theta(t) = \pi/4$ for $t \geq T$?

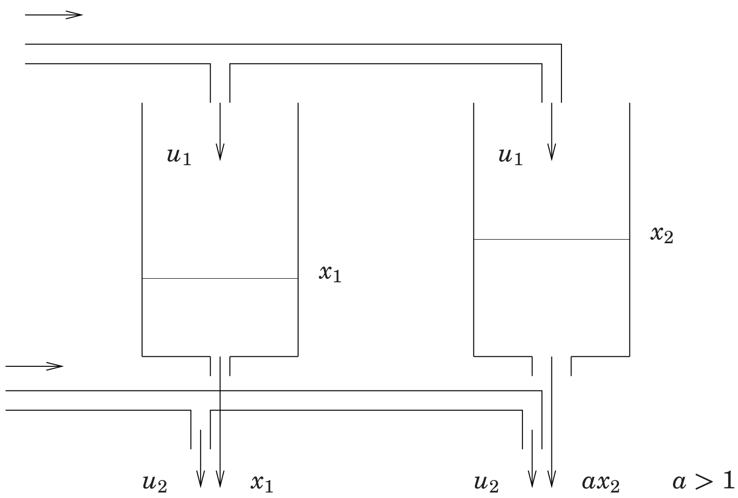


Figure 6.2 Example: Two water tanks

The first question turns out to be related to the concept of controllability, while the second relates to existence of a multivariable plant zero. □

The ball in the hoop is a single-input–single-output system. In the next example, we will ask analogous questions for a multivariable system.

Example 2 The water levels in two tanks in Fig. 6.2 are varying according to the equations

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + u_1(t)\\ \dot{x}_2(t) &= -ax_2(t) + u_1(t)\end{aligned}$$

and the measured output flows are affected by an additional input

$$\begin{aligned}y_1(t) &= x_1(t) + u_2(t)\\ y_2(t) &= ax_2(t) + u_2(t)\end{aligned}$$

For this system we are asking the questions

1. Starting from stationary water levels at nominal positions ($\dot{x}_1(0) = x_1(0) = 0$ and $\dot{x}_2(0) = x_2(0) = 0$), is it possible to bring the output flow to $y_1(t) = 1$, $y_2(t) = 2$ at time $t = T$ using the inputs $u_1(t)$ and $u_2(t)$ for $0 \le t < T$?
2. Using the same two inputs, is it possible to keep the outputs at $y_1(t) = 1$, $y_2(t) = 2$ for $t \ge T$?

□

6.2 Controllability

To study the first question of the examples, we make the following definition.

Definition 6.1
The system $\dot{x}(t) = Ax(t) + Bu(t)$ is said to be *controllable* , if for every $x_1 \in \mathbb{R}^n$ there exists $u(t), t \in [0, t_1]$, such that $x(t_1) = x_1$ is reached from $x(0) = 0$. The collection of vectors x_1 that can be reached in this way is called the *controllable subspace*.

If A is stable (has all poles in the left half-plane), define the *controllability Gramian*

$$W_c = \int_0^\infty e^{At} BB^T e^{A^T t} dt$$

□

There are several criteria for controllability:

PROPOSITION 6.1
The following statements regarding a system $\dot{x}(t) = Ax(t) + Bu(t)$ of order n are equivalent:

(i) The system is controllable

(ii) rank $[A - \lambda I \quad B] = n$ for all $\lambda \in \mathbb{C}$

(iii) rank $[B \; AB \dots A^{n-1}B] = n$

For stable systems there is a fourth statement equivalent to (i) -(iii):

(iv) The controllability Gramian is non-singular

□

The controllability Gramian measures gives a quantitative measure of how difficult it is in a stable system to reach a certain state:

THEOREM 6.1
Let $W_{c1} = \int_0^{t_1} e^{At} BB^T e^{A^T t} dt$. Then, for the system $\dot{x}(t) = Ax(t) + Bu(t)$ to reach $x(t_1) = x_1$ from $x(0) = 0$ it is necessary that

$$\int_0^{t_1} |u(t)|^2 dt \geq x_1^T W_{c1}^{-1} x_1 \geq x_1^T W_c^{-1} x_1$$

where W_c is the controllability Gramian. The first inequality becomes equality for

$$u(t) = B^T e^{A^T (t_1 - t)} W_{c1}^{-1} x_1$$

□

Proof. The matrix inequality $W_{c1} \leq W_c$ holds by definition.

$$\begin{aligned}
0 &\leq \int_0^{t_1} [x_1^T W_{c1}^{-1} e^{A(t_1-t)} B - u(t)^T][B^T e^{A^T(t_1-t)} W_{c1}^{-1} x_1 - u(t)] dt \\
&= x_1^T W_{c1}^{-1} \int_0^{t_1} e^{At} BB^T e^{A^T t} dt \; W_{c1}^{-1} x_1 \\
&\quad - 2x_1^T W_{c1}^{-1} \int_0^{t_1} e^{A(t_1-t)} Bu(t) dt + \int_0^{t_1} |u(t)|^2 dt \\
&= -x_1^T W_{c1}^{-1} x_1 + \int_0^{t_1} |u(t)|^2 dt
\end{aligned}$$

so $\int_0^{t_1} |u(t)|^2 dt \geq x_1^T W_{c1}^{-1} x_1$ with equality attained for $u(t) = B^T e^{A^T(t_1-t)} W_{c1}^{-1} x_1$. This completes the proof. □

The controllability Gramian can be computed by solving a linear system of equations:

THEOREM 6.2
Given that all eigenvalues of the matrix A have negative real parts, the matrix $W_c = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ satisfies the Lyapunov equation

$$AW_c + W_cA^T + BB^T = 0$$

□

Proof. A change of integration variable gives

$$\int_h^\infty e^{At}BB^Te^{A^Tt}dt = \int_0^\infty e^{A(t+h)}BB^Te^{A^T(t+h)}dt$$

Differentiating both sides with respect to h gives

$$\begin{aligned}
-e^{Ah}BB^Te^{A^Th} &= \frac{d}{dh}\int_h^\infty e^{At}BB^Te^{A^Tt}dt \\
&= \int_0^\infty \frac{d}{dh}[e^{A(t+h)}BB^Te^{A^T(t+h)}]dt \\
&= \int_0^\infty [Ae^{A(t+h)}BB^Te^{A^T(t+h)} + e^{A(t+h)}BB^Te^{A^T(t+h)}A^T]dt \\
&= A\left(\int_0^\infty e^{A(t+h)}BB^Te^{A^T(t+h)}dt\right) + \left(\int_0^\infty e^{A(t+h)}BB^Te^{A^T(t+h)}dt\right)A^T
\end{aligned}$$

Inserting $h = 0$ yields

$$-BB^T = AW_c + W_cA^T$$

□

Example 3 Two address the first question raised in Example 6.1, note that the dynamics has the form $\dot{x}(t) = Ax(t) + Bu(t)$ with

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -a \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

so the controllability Gramian is

$$W_c = \int_0^\infty \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}\begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$$

The matrix is non-singular when $a \neq 1$, so in this case the states and outputs to arbitrary positions and the answer to the first question is yes. On the other hand, if $a = 1$ the two tanks move identically and it is impossible to achieve $y_1 = 1$ and $y_2 = 2$ simultaneously.

It is natural to expect that the control problem becomes harder as a gets closer to 1. This is indeed true. In fact, when $a \approx 1$ the inverse Gramian W_c^{-1} has large entries, so the inequality

$$\int_0^\tau |u(t)|^2dt \geq \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix}^T W_c^{-1} \begin{bmatrix} x_1(\tau) \\ x_2(\tau) \end{bmatrix}$$

shows that some final states can be reached only with very large inputs. This is also intuitively clear, since the tanks then have similar dynamcs and it is difficult to make x_1 and x_2 move in opposite directions. □

6.3 Observability

The notion of controllability has a natural dual:

Definition 6.2
The system

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cy(t)$$

is *observable* if the initial state $x(0) = x_0 \in \mathbb{R}^n$ is uniquely determined by the output $y(t), t \in [0, t_1]$. The collection of vectors x_0 that cannot be distinguished from $x = 0$ is called the *unobservable subspace*.

If A is stable, the *observability Gramian* is defined as

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

□

There are several criteria for verification of observability:

Proposition 6.2
The following statements regarding a system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ of order n are equivalent:

(i) The system is observable

(ii) rank $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$

(iii) rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

For stable systems there is a fourth equivalent statement:

(iv) The observability Gramian is non-singular

□

The observability Gramian measures how difficult it is in a stable system to distinguish two initial states from each other by observing the output.

Theorem 6.3
Let $W_{o1} = \int_0^{t_1} e^{A^T t} C^T C e^{At} dt$. Then, for $\dot{x}(t) = Ax(t)$, the influence from the initial state $x(0) = x_0$ on the output $y(t) = Cx(t)$ satisfies

$$\int_0^{t_1} |y(t)|^2 dt = x_0^T W_{o1} x_0 \leq x_0^T W_o x_0$$

where W_o is the observability Gramian. □

Also the observability Gramian can be computed by solving a linear system of equations:

THEOREM 6.4
Given that all eigenvalues of the matrix A have negative real part, the matrix $W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ satisfies the Lyapunov equation

$$A^T W_o + W_o A + C^T C = 0$$

□

Proof. The result follows directly from the corresponding formula for the controllability Gramian by replacing A^T by A (which has no effect on the eigenvalues) and C by B^T. □

6.4 Poles and zeros

After Laplace transformation of the signals, the input-output relationship of linear time-invariant system can be written

$$Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{G(s)} U(s)$$

The transfer matrix $G(s)$ is rational provided that the system is of finite order. The points $p \in \mathbb{C}$ where $G(s) = \infty$ are called *poles of* G. They are eigenvalues of A and determine stability. The points $z \in \mathbb{C}$ where $G(s)$ loses rank are called *(transmission) zeros* of G. If $G(s)$ is square, the zeros are given by the roots to the numerator polynomial of $\det G(s)$. The following definitions can be used even when $G(s)$ is not a square matrix:

- A *pole of* G is a root of the *pole polynomial*, the least common denominator of all minors (sub-determinants) to $G(s)$. The multiplicity of the root determines the multiplicity of the pole.
- A *zero of* G is a root of the *zero polynomial*, the greatest common divisor to the numerators of the maximal minors in $G(s)$. The multiplicity of the root determines the multiplicity of the zero. When $G(s)$ is square, the zero polynomial is numerator of $\det G(s)$.

Recall that a pole p is an eigenvalue of A and has an associated state trajectory $x(t) = x_0 e^{pt}$. A zero describes how inputs and outputs couple to each other. In particular, a zero z means that an input $u(k) = u_0 e^{zt}$ is blocked. See Figure 6.3.

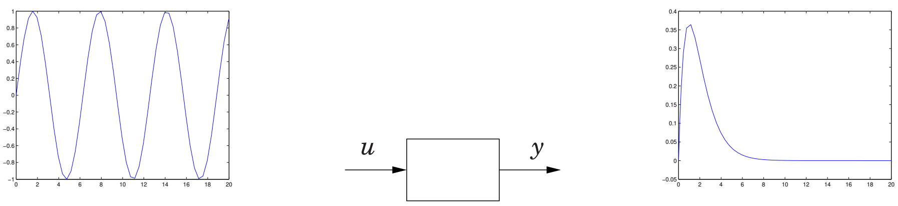


Figure 6.3 A pair of zeros at $\pm i\omega$ blocks inputs of that frequency.

Example 4 Recall the ball in the hoop of Example 6.1. The system equation $\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$ shows that The transfer function from ω to θ is

$$\frac{s}{s^2 + cs + k}$$

The zero in 0 makes it impossible to control the stationary position of the ball. In particular, the second question raised in the introduction has a negative answer. □

Example 5 Laplace transformation of the equations in Example 6.1 gives

$$sX_1 = -X_1 + U_1 \qquad Y_1(s) = X_1(s) + U_2(s) = \frac{1}{s+1}U_1(s) + U_2(s)$$

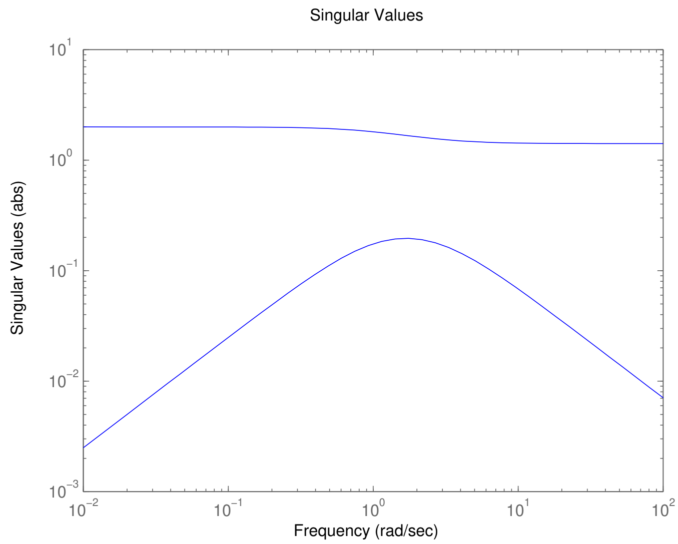
$$sX_2 = -2X_2 + U_1 \qquad Y_2(s) = 2X_2(s) + U_2(s) = \frac{1}{s+2}U_1(s) + U_2(s)$$

This gives the transfer matrix

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix} \qquad \det G(s) = \frac{-s}{(s+1)(s+2)}$$

so this system also has a zero in the origin, indicating that the stationary levels of the outputs cannot be controlled arbitrarily. In fact, for stationary inputs and states, the first state equation gives $u_1 = x_1$ and the second gives $u_1 = ax_2$, so the two outputs must be equal.

Plotting the singular values of $G(i\omega)$ gives



The largest singular value is fairly constant. This is due to the second input. The first input makes it possible to control the difference between the two tanks, but mainly near $\omega = 1$ where the dynamics make a difference. □

6.5 State-space realizations in diagonal form

Consider a transfer matrix with partial fraction expansion

$$G(s) = \sum_{i=1}^{n} \frac{C_i B_i}{s - p_i} + D$$

This has the realization

$$\dot{x}(t) = \begin{bmatrix} p_1 I & & 0 \\ & \ddots & \\ 0 & & p_n I \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(t)$$

$$y(t) = [\, C_1 \quad \dots \quad C_n \,]\, x(t) + Du(t)$$

The rank of the matrix $C_i B_i$ determines the necessary number of rows in B_i, the number of columns in C_i and the multiplicity of the pole p_i.

Example 6 The system

$$G(s) = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{2}{(s+1)(s+3)} \\ \dfrac{6}{(s+2)(s+4)} & \dfrac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{s+1} & \dfrac{1}{s+1} - \dfrac{1}{s+3} \\ \dfrac{3}{s+2} - \dfrac{3}{s+4} & \dfrac{1}{s+2} \end{bmatrix}$$

$$= \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix}}{s+2} - \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}}{s+3} - \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \end{bmatrix}}{s+4}$$

has the realization

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$

□