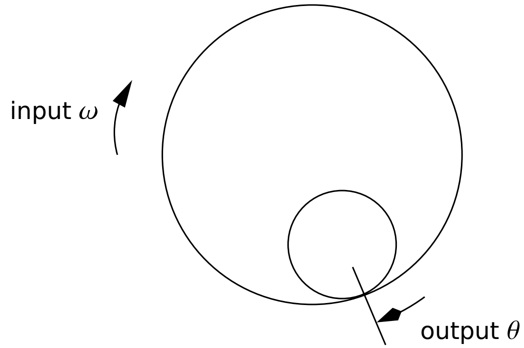
Lecture 6 – Outline

1. Controllability and observability, Gramians
2. Multivariable poles and zeros
3. Minimal realizations

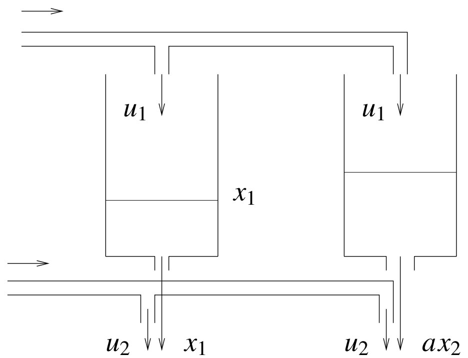
Example: Ball in the Hoop



$$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$$

Can you reach $\theta = \pi/4$, $\dot{\theta} = 0$? Can you stay there?

Example: Two water tanks



$a \geq 1$

$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$
$$\dot{x}_2 = -ax_2 + u_1 \qquad y_2 = ax_2 + u_2$$

Can you reach $y_1 = 1, y_2 = 2$? Can you stay there?

Lecture 6 – Outline

1. Controllability and observability
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Review: State feedback and controllability

Process

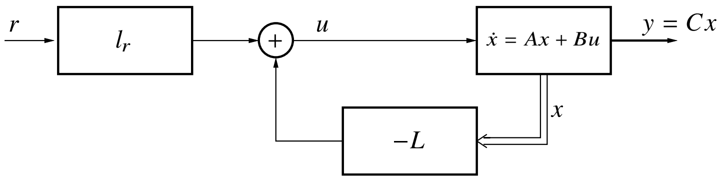
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Closed-loop system

$$\begin{cases} \dot{x} = (A - BL)x + Bl_r r \\ y = Cx \end{cases}$$

State-feedback control

$$u = -Lx + l_r r$$



If the system (A, B) is *controllable* then we can place the eigenvalues of $(A - BL)$ arbitrarily

Review: State observers and observability

Process

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Observer ("Kalman filter")

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

Estimation/observer error $\tilde{x} = x - \hat{x}$:

$$\dot{\tilde{x}} = (A - KC)\tilde{x}$$

If the system (A, C) is *observable* then we can place the eigenvalues of $(A - KC)$ arbitrarily

Controllability – definition

The system

$$\dot{x} = Ax + Bu$$

is **controllable**, if for every $x_1 \in \mathbb{R}^n$ there exists $u(t)$, $t \in [0,\ t_1]$, such that $x(t_1) = x_1$ can be reached from $x(0) = 0$.

The collection of vectors x_1 that can be reached in this way is called the **controllable subspace**.

Controllability criteria

The following controllability criteria for a system $\dot{x} = Ax + Bu$ of order n are equivalent:

(i) rank $[B\ AB \ldots A^{n-1}B] = n$

(ii) rank $[\lambda I - A \quad B] = n$ for all $\lambda \in \mathbb{C}$

If the system is stable, define the **controllability Gramian**

$$W_c = \int_0^\infty e^{At}BB^T e^{A^T t}\,dt$$

For such systems there is a third equivalent criterion:

(iii) The controllability Gramian is non-singular

Interpretation of the controllability Gramian

The inverse of the controllability Gramian measures how difficult it is to reach different states.

In fact, the minimum control energy required to reach $x = x_1$ starting from $x = 0$ satisfies

$$\int_0^\infty |u(t)|^2 dt = x_1^T W_c^{-1} x_1$$

(For proof, see the lecture notes.)

Computing the controllability Gramian

The controllability Gramian $W_c = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ can be computed by solving the Lyapunov equation

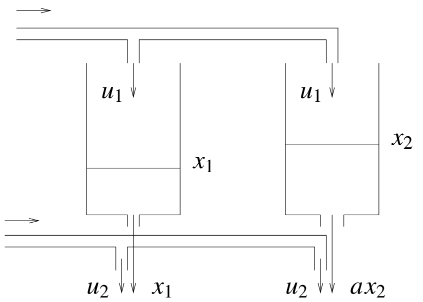
$$A W_c + W_c A^T + B B^T = 0$$

(For proof, see the lecture notes.)

(Matlab: `Wc = lyap(A,B*B')`)

Q: Where have we seen this equation before?

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -a x_2 + u_1$$

Controllability Gramian: $W_c = \int_0^\infty \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$

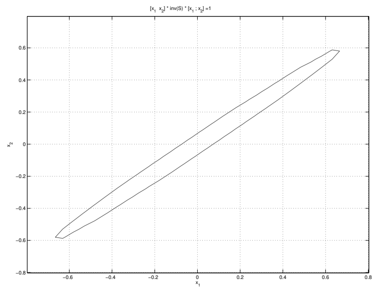
W_c close to singular when $a \approx 1$. Interpretation?

Example cont'd

Matlab:

```
>> a = 1.25;  A = [-1 0; 0 -1*a];  B = [1; 1];
```

```
>> CM = [B  A*B], rank(CM)
CM =
    1.0000   -1.0000
    1.0000   -1.2500
ans =
     2
>> Wc = lyap(A,B*B')
Wc =
   0.5000    0.4444
   0.4444    0.4000
>> invWc = inv(Wc)
invWc =
    162.0   -180.0
   -180.0    202.5
```



Plot of $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot W_c^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$ corresponds to the states we can reach by $\int_0^\infty |u(t)|^2 dt = 1$.

Observability – definition

The system

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

is **observable**, if the initial state $x(0) = x_0 \in \mathbb{R}^n$ can be uniquely determined by the output $y(t)$, $t \in [0,\, t_1]$.

The collection of vectors x_0 that cannot be distinguished from $x = 0$ is called the **unobservable subspace**.

Observability criteria

The following observability criteria for a system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ of order n are equivalent:

(i) rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

(ii) rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$

If the system is stable, define the **observability Gramian**

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

For such systems there is a third equivalent statement:

(iii) The observability Gramian is non-singular

Interpretation of the observability Gramian

The observability Gramian measures how easy it is to distinguish an initial state from zero by observing the output.

In fact, the influence of the initial state $x(0) = x_0$ on the output $y(t)$ satisfies

$$\int_0^\infty |y(t)|^2 dt = x_0^T W_o x_0$$

Computing the observability Gramian

The observability Gramian $W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ can be computed by solving the Lyapunov equation

$$A^T W_o + W_o A + C^T C = 0$$

(Matlab: `Wo = lyap(A',C'*C)`)

Mini-problem

Is the water tank system with $a = 1$ observable?

What if only y_1 is available?

Lecture 6 – Outline

1. Controllability and observability
2. Multivariable poles and zeros
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Poles and zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{G(s)} U(s)$$

For **scalar** systems,

- the points $p \in \mathbb{C}$ where $G(p) = \infty$ are called **poles**
- the points $z \in \mathbb{C}$ where $G(z) = 0$ are called **zeros**

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Poles and zeros

For multivariable systems,

- the points $p \in \mathbb{C}$ where any $G_{ij}(p) = \infty$ are called **poles**
- the points $z \in \mathbb{C}$ where $G(z)$ loses rank are called **(transmission) zeros**

Example:

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: -2 and -1 (but what about their multiplicity?)

Zeros: 1 (but how to find them?)

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Pole and zero polynomials

- The **pole polynomial** is the least common denominator of all minors* of $G(s)$.
- The **zero polynomial** is the greatest common divisor of the maximal minors of $G(s)$, normalized to the have the pole polynomial as denominator.

The **poles** of G are the roots of the pole polynomial.

The **(transmission) zeros** of G are the roots of the zero polynomial.

* A minor of a matrix A is the determinant of some square submatrix, obtained by removing zero or more of A's rows and columns

Poles and zeros – example

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: Minors: $\frac{2}{s+1}, \frac{3}{s+2}, \frac{1}{s+1}, \frac{1}{s+1}, \frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{-(s-1)}{(s+1)^2(s+2)}$

The least common denominator is $(s+1)^2(s+2)$, giving the poles -2 (with multiplicity 1) and -1 (with multiplicity 2)

Zeros: Maximal minor: $\frac{-(s-1)}{(s+1)^2(s+2)}$ (already normalized)

The greatest common divisor is $s-1$, giving the (transmission) zero 1 (with multiplicity 1)

(Matlab: `tzero(G)`)

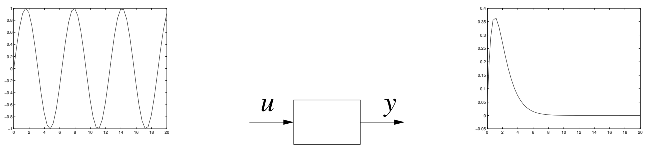
Interpretation of poles and zeros

Poles:

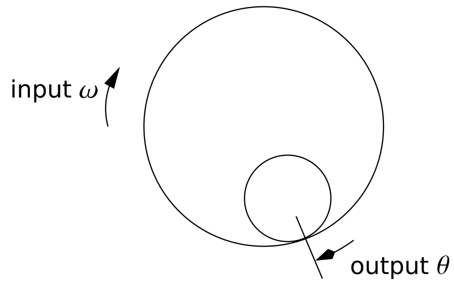
- A pole p is associated with the state response $x(t) = x_0 e^{pt}$
- A pole p is an eigenvalue of A

Zeros:

- A zero z means that an input $u(t) = u_0 e^{zt}$ is blocked
  - For a multivariable system, blocking occurs only in a certain input direction
- A zero describes how inputs and outputs couple to states



Example: Ball in the Hoop

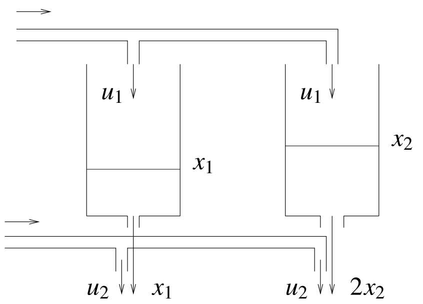


$$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$$

The transfer function from ω to θ is $\frac{s}{s^2+cs+k}$. The zero in $s=0$ makes it impossible to control the stationary position of the ball.

- Zeros are not affected by feedback!

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$
$$\dot{x}_2 = -2x_2 + u_1 \qquad y_2 = 2x_2 + u_2$$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix} \qquad \det G(s) = \frac{-s}{(s+1)(s+2)}$$

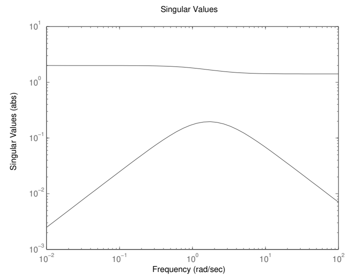
The system has a zero in the origin! At stationarity $y_1 = y_2$.

Plot singular values of $G(i\omega)$ vs frequency

```
» s=tf('s')
» G=[1/(s+1) 1 ; 2/(s+2) 1]
» sigma(G) ; plot singular values

% Alt. for a certain frequency:

» w=1;
» A = freqresp(G,i*w);
» [U,S,V] = svd(A)
```



The largest singular value of $G(i\omega) = \begin{bmatrix} \frac{1}{i\omega+1} & 1 \\ \frac{2}{i\omega+2} & 1 \end{bmatrix}$ is fairly constant.

This is due to the second input. The first input makes it possible to control the difference between the two tanks, but mainly near $\omega = 1$ where the dynamics make a difference.

Lecture 6 – Outline

1. Controllability and observability
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Minimal realization – definition

Given $G(s)$, any state-space model (A, B, C, D) that is both **controllable** and **observable** and has the same input–output behavior as $G(s)$ is called a **minimal realization**.

A transfer function with n poles (counting multiplicity) has a minimal realization of order n.

Realization in diagonal form

Consider a transfer function with partial fraction expansion

$$G(s) = \sum_{i=1}^{n} \frac{C_i B_i}{s - p_i} + D$$

This has the realization

$$\dot{x}(t) = \begin{bmatrix} p_1 I & & 0 \\ & \ddots & \\ 0 & & p_n I \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix} x(t) + Du(t)$$

The rank of the matrix $C_i B_i$ determines the necessary number of columns in B_i and the multiplicity of the pole p_i.

Realization of multivariable system – example 1

To find a minimal realization for the system

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

with poles in -2 and -1 (double), write the transfer matrix as (e.g.)

$$G(s) = \frac{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+2}$$

giving the realization

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix} x$$

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Realization of multivariable system – example 2

To find state space-realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+3)} \\ \frac{6}{(s+2)(s+4)} & \frac{1}{s+2} \end{bmatrix}$$

write the transfer matrix as

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+3} \\ \frac{3}{s+2} - \frac{3}{s+4} & \frac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix}}{s+2} + \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \end{bmatrix}}{s+3} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \end{bmatrix}}{s+4}$$

This gives the realization

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$

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Summary

- Gramians give quantitative answers to how controllable or observable a system is in different state directions
  - Warning: They do not reveal some important frequency-domain information (see next lecture)
- A multivariable zero blocks input signals a certain direction
- A minimal state-space realization describes the controllable and observable subspace of a system

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