Lecture 4

Control Synthesis in the Frequency Domain*

Recall from the first chapter the illustration of the design process shown in Figure 4.1. While previous lectures were mainly concerned with analysis, modeling and specifications, we will now focus on the synthesis problem for scalar processes. Another name for such systems is single-input-single-output (SISO) systems. In the presentation, we separate the solution into feedback control and feedforward control.

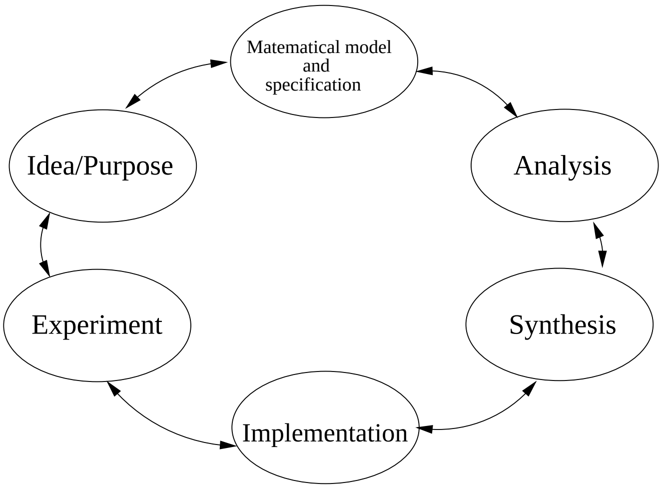

Figure 4.1 Schematic overview of the design process

4.1 Frequency domain specifications

From Lecture 3 we recall that the following relations hold between the Laplace transforms of the signals in the closed-loop system in Figure 4.2.

$$
\begin{aligned}
Z &= \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R \\
Y &= \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R \\
U &= -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R
\end{aligned}
$$

Among these relations we find the *sensitivity function*, $S = (1+PC)^{-1}$, and the *complementary sensitivity function*, $T = (1+PC)^{-1}PC$. A good control design should fulfill several properties:

Small influence of load disturbances, implying that $(1+PC)^{-1}P = PS$ should be small. The requirement is especially important for low frequencies (including zero), while high-frequency disturbances are naturally filtered out by the process itself.

Limited amplification of measurement noise, implying that $(1+PC)^{-1}C = CS$ should be small. Since measurement noise is often dominated by high frequencies the requirement is especially important in the upper frequency range.

*Written by A. Rantzer with contributions by K.J. Åström

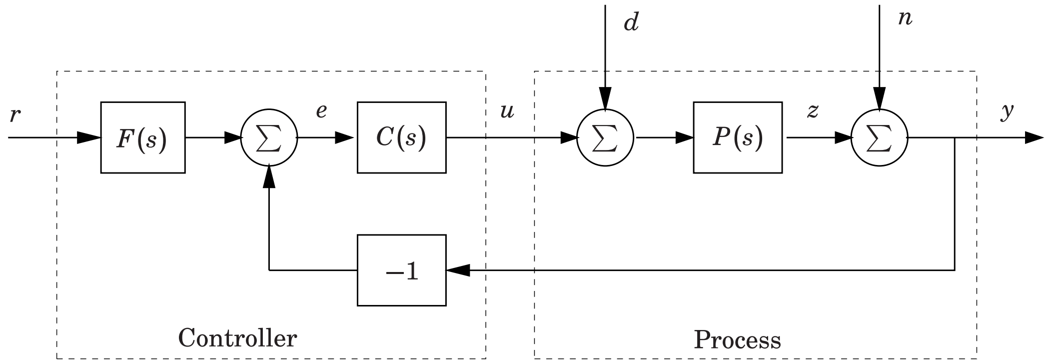

Figure 4.2 A controller with two degrees of freedom

Small influence of model errors on closed-loop system, implying that the sensitivity function S should be small.

Robust stability despite model errors, implying that T should be small according to the small gain analysis presented in Lecture 2. The requirement is especially important for high frequencies, where models tend to be less accurate.

Accurate tracking of the setpoint, implying that the transfer function $(1+PC)^{-1}PCF = TF$ should be close to 1 up to some desired bandwidth ω_b of the closed-loop system.

All of the above requirements are of course impossible to fulfill for all frequencies. A typical design compromise is therefore done as follows: The sensitivity S is designed to be small (approach 0) for low frequencies. This takes care of disturbance rejection and sensitivity towards modeling errors up to the cross-over frequency of the open loop. At the same time, $T \approx 1$ for low frequencies, implying good reference tracking if F is suitably designed (see the last part of this lecture). For high frequencies, the complementary sensitivity T is designed to be small (approach 0), which takes care of noise and model errors in the upper frequency range.

Design specifications on S and T (and possibly also on CS and PS) can be expressed in different ways. For S and T it is common to state their maximum allowed magnitudes, M_s and M_t, since they are closely related to the stability margins and robustness of the system. Typical specifications on M_s and M_t are in the range 1.4–2.0, where smaller values indicate a more robust system. A more elaborate way to express frequency-domain specifications is in the form of weighting functions $W_T(s)$ and $W_S(s)$:

$$\begin{aligned}
\|W_S S\|_\infty \le 1 \quad &\Leftrightarrow \quad |S(i\omega)| \le |W_S^{-1}(i\omega)|, \ \forall\omega \\
\|W_T T\|_\infty \le 1 \quad &\Leftrightarrow \quad |T(i\omega)| \le |W_T^{-1}(i\omega)|, \ \forall\omega
\end{aligned}$$

(This notation is common in so called $\mathcal{H}_\infty$ synthesis methods.) It can also be useful to state piecewise specifications for different frequency intervals.

4.2 Loop shaping

From the closed-loop transfer functions studied above we can note that the closed-loop performance depends critically on the *loop transfer function*

$$L(s) = P(s)C(s)$$

In particular, the sensitivity functions can be written as $S = (1 + L)^{-1}$ and $T = L(1 + L)^{-1}$ respectively. A popular approach to control synthesis, known as *loop shaping*, is to focus on

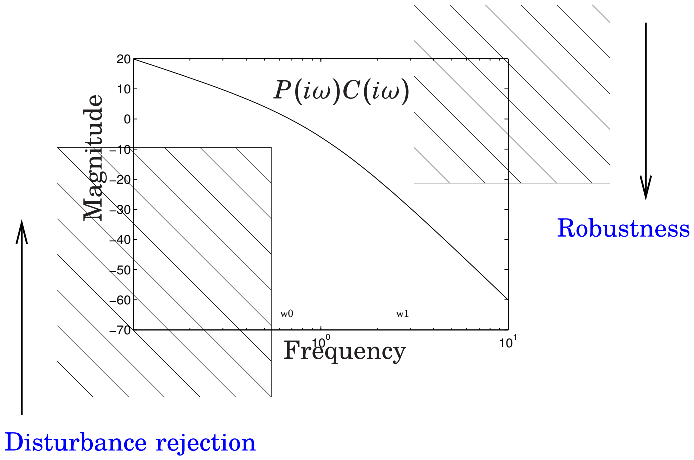

Figure 4.3 Frequency specifications for the closed-loop system; for the sensitivity function $S = \frac{1}{1+L}$ and for the complementary sensitivity function $T = \frac{L}{1+L}$, can (approximately) be interpreted as frequency specifications in open loop for the *loop transfer matrix* $L = P(i\omega)C(i\omega)$, such that L should have small norm $\|P(i\omega)C(i\omega)\|$ at high frequencies, while at low the frequencies instead $\|[P(i\omega)C(i\omega)]^{-1}\|$ should be small.

the shape of the loop transfer function and keep modifying $C(s)$ until the desired shape is obtained. We have seen that that proper disturbance rejection requires small sensitivity S (large L) for for small frequencies, while process uncertainty requires the complementary sensitivity function to be small (small L) for high frequencies, see Fig. 4.3. On the other hand, if the amplitude of L decreases very rapidly, the phase tends to become lower than $-180°$ and the system becomes unstable. Loop shaping is therefore a trade-off between different kinds of specifications.

Many control problems can be adequately solved by PID controllers, which can be viewed a combination of one lag compensator and one lead compensator. For more advanced applications, like resonant systems, higher order controllers are desirable. An example of such a system is the flexible servo treated in lab 1.

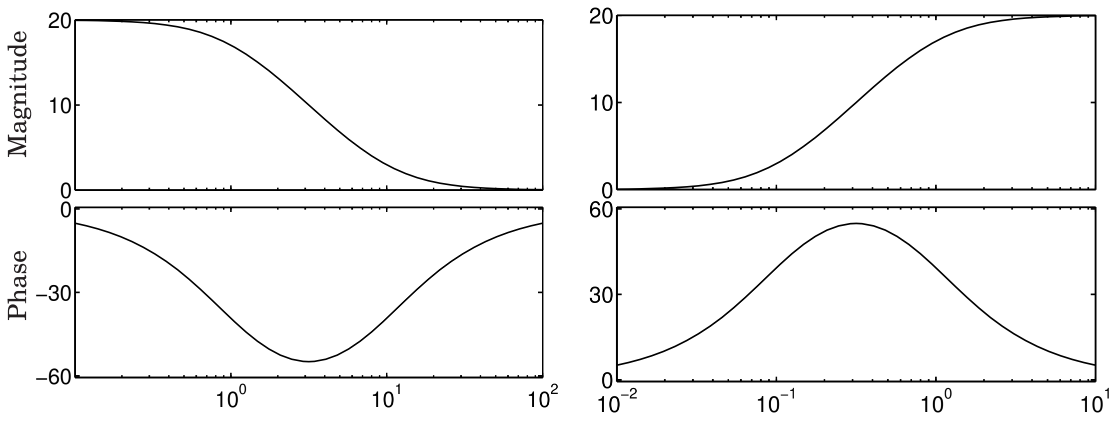

Figure 4.4 Bode diagrams for lag compensator $\frac{s+10}{s+1}$ (left) and lead compensator $\frac{10s+1}{s+1}$ (right)

Graphical illustrations in Bode or Nichols diagrams are typically used to support the design. These diagrams are convenient because of the logarithmic scale, where the controller contributes additively to the loop transfer function:

$$\log|L(i\omega)| = \log|P(i\omega)| + \log|C(i\omega)|$$
$$\arg|L(i\omega)| = \arg P(i\omega) + \arg C(i\omega)$$

From the basic course, recall the following essential properties of lead and lag compensators, illustrated in Figure 4.4:

Lag compensator

- Increases low frequency gain: Can be used to reduce stationary errors
- Decreases phase, which may reduce stability margins

Lead compensator

- Increases high frequency gain: Can be used for faster closed-loop response
- Increases phase, which may improve stability margins

Loop shaping design of high-order controllers will be exercised in lab 1. We will first design a controller $C_1(s)$ for low frequencies, then keep adding compensator links $C_2(s), C_3(s), \dots$ to modify the dynamics at higher and higher frequencies until a satisfactory controller $C(s) = C_1(s) \cdots C_m(s)$ is obtained. Lead/lag links are often sufficient, but occasionally it is useful to also consider controllers with poles or zeros outside the real axis. The figure below shows the Bode diagram for cases with stable complex zeros (left) and complex poles (right).

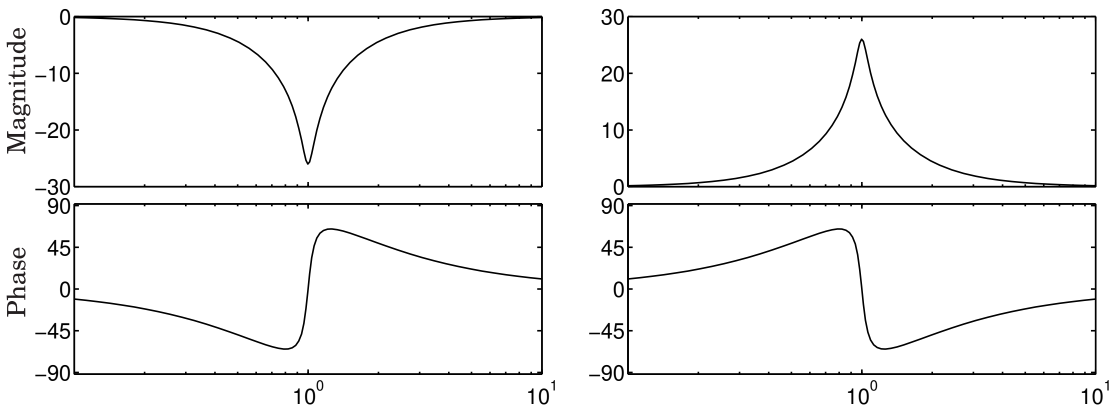

Figure 4.5 Notch compensator $\frac{s^2+0.1s+1}{(s+1)^2}$ (left) and resonant compensator $\frac{(s+1)^2}{s^2+0.1s+1}$ (right)

4.3 Feedforward synthesis

Let us finally consider the design of $F(s)$ to shape the response to reference signals. See Figure 4.6. The usual interpretation of the reference signal r is that it specifies the desired value of y. Hence the transfer function from r to y should be as close to identity as possible, ideally

$$\frac{P(s)C(s)}{1+P(s)C(s)}F(s) = 1 \quad \text{or equivalently} \quad F(s) = \frac{1+P(s)C(s)}{P(s)C(s)}$$

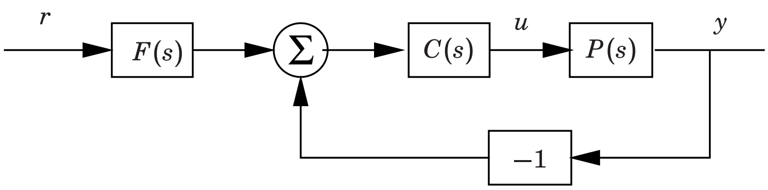

Figure 4.6 The feedforward filter $F(s)$ is used to improve the response to reference signals

Unfortunately, this is impossible to achieve because $PC/(1+PC)$ generally has more poles than zeros (pole excess) and hence $F(s)$ would not be proper.

Instead, $F(s)$ is usually chosen to approximate $(1+PC)/(PC)$ at small frequencies. The simplest (and most common) choice is to make F constant and equal to

$$\frac{1+P(0)C(0)}{P(0)C(0)}$$

A more advanced option is to choose

$$\frac{P(s)C(s)}{1+P(s)C(s)}F(s) = \frac{1}{(sT+1)^d}$$

for some suitable time constant T and with d large enough to make F proper and implementable.

Example 3 If

$$P(s) = \frac{1}{(s+1)^4} \qquad\qquad \frac{P(s)C(s)}{1+P(s)C(s)}F(s) = \frac{1}{(sT+1)^4}$$

then the closed-loop transfer function from r to u becomes

$$\frac{C(s)}{1+P(s)C(s)}F(s) = \frac{(s+1)^4}{(sT+1)^4}$$

The gain is 1 for low frequencies ($s \approx 0$) but $1/T^4$ for $s = 1$. Hence, fast response (T small) requires high controller gain. Bounds on the control signal therefore limit how fast response we can obtain. □