Lecture 3

Specifications and disturbance models*

This lecture has two parts. In the first part, we will review some overall design specifications for simple control loops. In the second part, we will study how to model process and measurement disturbances as stochastic processes.

3.1 Specifications

Let us discuss some specifications we typically impose on control loops, beyond the obvious requirement of closed-loop stability. For simplicity, we will restrict our attention to the basic two-degrees-of-freedom structure shown in Figure 3.1, where we assume a scalar transfer function $P(s)$ for the plant. This setup was studied in the basic course and is sufficient for many practical situations.

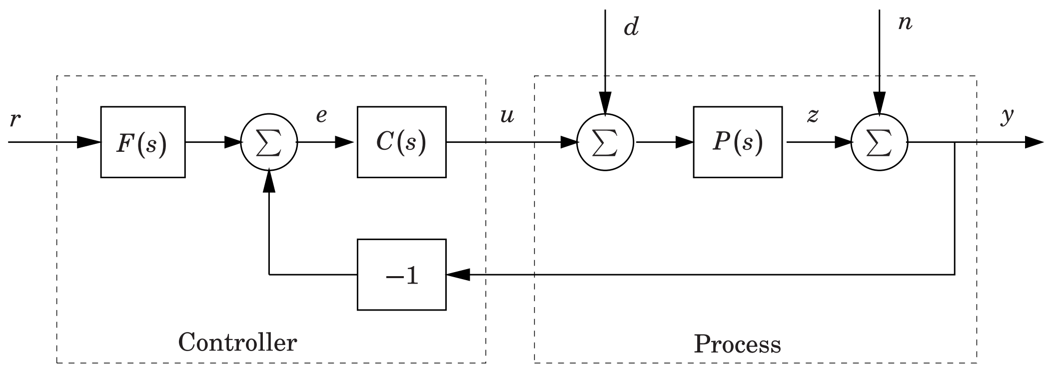

Figure 3.1 A controller with two degrees of freedom

The controller consists of two transfer functions, the feedback part $C(s)$ and the feedforward part $F(s)$. The control objective is to keep the process output z close to the reference signal r, in spite of load disturbances d. The measurement y is corrupted by noise n.

Several types of specifications could be relevant for this control loop.

A: Reduce the effects of load disturbances

B: Limit the effects of measurement noise

C: Reduce sensitivity to process variations

D: Make the output follow command signals

A useful synthesis approach is to first design $C(s)$ to meet the specifications A, B, and C, then design $F(s)$, to deal with the response to reference changes, D. However, the two steps are

*Written by A. Rantzer with contributions by K.J. Åström not completely independent: A poor feedback design will have a negative influence also on the response to reference signals.

The following relations hold between the Laplace transforms of the signals in the closed-loop system:

$$
\begin{aligned}
Z &= \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R \\
Y &= \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R \\
U &= -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R
\end{aligned}
$$

Several observations can be made:

- The signals in the feedback loop are characterized by four transfer functions (sometimes called the "Gang of Four"):

$$
\frac{1}{1+P(s)C(s)} \qquad \frac{P(s)}{1+P(s)C(s)} \qquad \frac{C(s)}{1+P(s)C(s)} \qquad \frac{P(s)C(s)}{1+P(s)C(s)}
$$

  In particular, we recognize the first one as the sensitivity function, $S(s)$, and the last one as the complementary sensitivity function, $T(s)$.
- The total system with a controller having two degrees of freedom is characterized by six transfer functions (the "Gang of Six").

To fully understand the properties of the closed-loop system, it is necessary to look at all the transfer functions. It can be strongly misleading to only show properties of a few input-output maps, for example only a step response from reference signal to process output. This is a common mistake in the literature.

The properties of the different transfer functions can be illustrated in several ways, by time or frequency responses. For a particular example, we show the six frequency response amplitudes in Figure 3.2 and the corresponding six step responses in Figure 3.3.

It is worthwhile to compare the frequency plots and the step responses and to relate their shape to the specifications A–D:

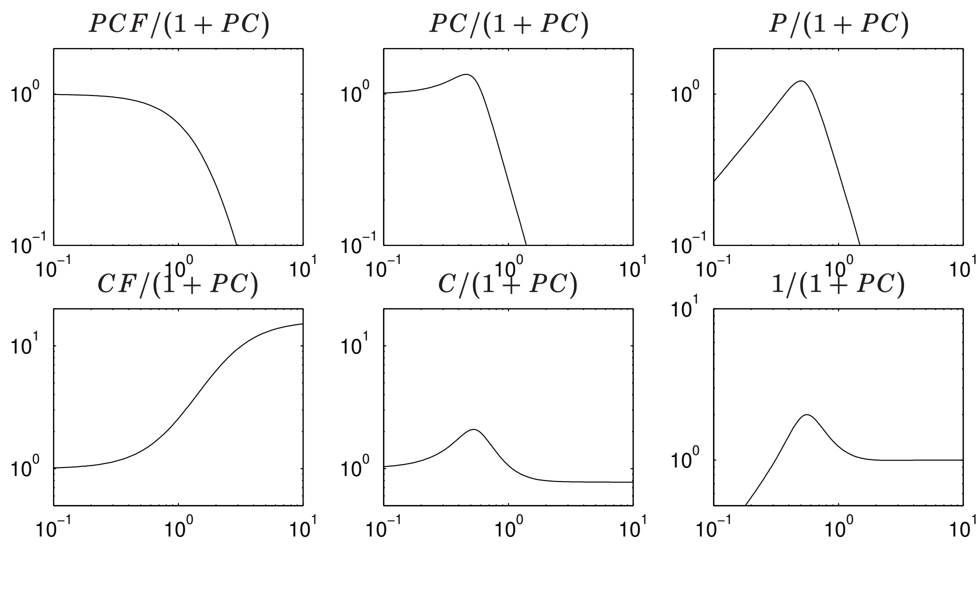

Figure 3.2 Frequency response amplitudes for $P(s) = (s+1)^{-4}$, $C(s) = 0.775(s^{-1}/2.05 + 1)$ when $F(s)$ is designed to give $PCF/(1+PC) = (0.5s+1)^{-4}$

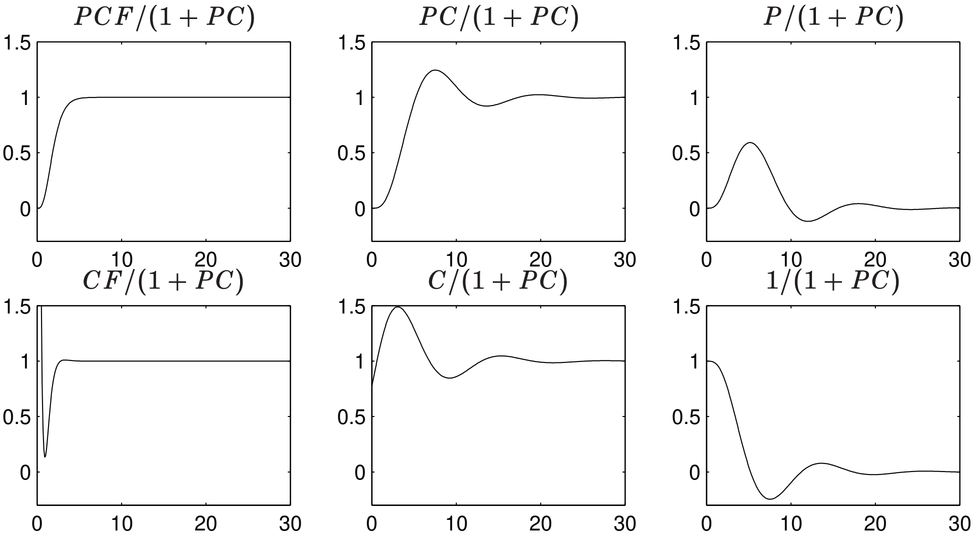

Figure 3.3 Step responses for $P(s) = (s+1)^{-4}$, $C(s) = 0.775(s^{-1}/2.05 + 1)$ when $F(s)$ is designed to give $PCF/(1+PC) = (0.5s+1)^{-4}$

Disturbance rejection The two upper right plots show the effect of the disturbance d in control signal u and process output z respectively. The resulting process error should not be too large and should settle to zero quickly enough. The control input would cancel the disturbance exactly if the mid upper step response would be an ideal step. In a short time-scale this is impossible, since the control input will not change until the effect of the disturbance has appeared in the process output and been available for measurement. However, slow disturbances should normally be cancelled by u. Equivalently, *the sensitivity function* $1/(1+PC)$ *should be small for low frequencies*. This specification is usually corresponds to an integrator in the controller.

Suppression of measurement noise The second specification was to limit the effect of measurement noise, typically a high frequency phenomenon. The mid upper frequency plot shows good attenuation of measurement noise above the "cut-off" frequency of 1 Hz. In this example, this is mainly an effect of the process dynamics. A more interesting question is maybe the gain from measurement noise to control input, since fast oscillations in the control actuator are usually undesirable. For this aspect, the mid lower frequency plot, showing the Bode amplitude from n to u, is of interest.

Robustness to process variations As shown in the previous lecture, the robustness to process variations is determined by the sensitivity functions. In this example, the lower right frequency plot has a maximal value of 2, which shows that a small relative error in the process can give rise to a relative error of double size in the closed-loop transfer function. The maximal amplitude of the frequency plot for the complementary sensitivity function is 1.35, so the small gain theorem proves stability of the closed-loop system as long as the relative error in the process model is below $74\% = 1/1.35$. In fact, most process models are inaccurate at high frequencies, so *the complementary sensitivity function* $PC/(1+PC)$ *should be small for high frequencies*.

Command response The upper left corner plot shows the map from reference signal r to process output z. Using the prefilter F, it is possible to get a better step response here than in the upper mid plot. The prize to pay is that the corresponding response in the control signal gets higher amplitude. This can be seen by comparing the lower left plot, showing the map from r to u, to the lower mid plot, which shows the corresponding map when $F \equiv 1$.

3.2 Disturbance models

This section reviews the main aspects in disturbance modelling and the corresponding relations of descriptions in the time and frequency domain, respectively.

We will also consider the two related questions:

(i) Given a known input spectra and known transfer function, what is the spectral density of the output

(ii) Given a known spectral density for a signal, find a stable linear system with *white noise* input which gives the same spectral density on its output.

The latter problem is called the *spectral factorization problem* and will be used to rewrite systems with coloured disturbances to an equivalent system with white noise input, which will be used as a standard form for different estimation and prediction problems later on in course.

In the basic control diagram of Fig. 3.4 we consider load disturbances d and measurement noise n

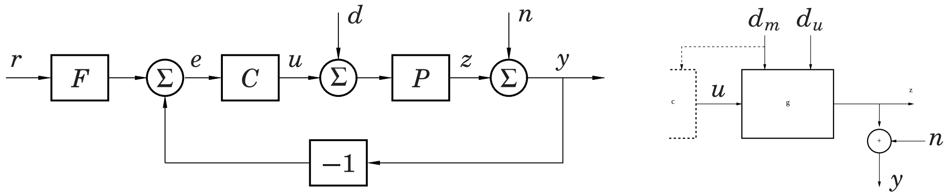

Figure 3.4 *(Left)* The basic control loop with load disturbances d and measurement noise n. *(Right)* Load disturbances which can be measured, d_m, e.g., changes in outer temperature, can be (partially) compensated for by feedforward to the control signal.

The load (or process) disturbance d drives the system from its desired state, whereas the measurement noise n corrupts the feedback information about z. Load disturbances can be divided into measurable disturbances, d_m, which partially can be compensated by feedforward, and load disturbances d_u that cannot be measured. Even if we cannot measure d_u in Fig. 3.4, statistical information like covariance or spectral density will help us to design controllers which reduces/supresses the effect of the disturbances with respect to, e.g., average and variance of the control objective z.

Example 1
In paper production there are a lot of disturbances which affect the paper quality and the paper thickness. One objective is to keep down the variation in the paper thickness, see Fig. 3.5. All paper production below the test limit is wasted. Good control allows for lower setpoint with the same yield. By having a lower variance of the production, the average paper thickness can thus also be lower, which saves significant costs in both energy and raw material. Keeping down the variance of the output will be an important control objective for us in this course. □

Stochastic processes
A **stochastic process** (random process, random function) is a family of stochastic variables $\{x(t), t \in T\}$ where t represents time. The stocastic process can be viewed as a function of two variables $x(t, \omega)$. For a fixed $\omega = \omega_0$ it gives a time function $x(\cdot, \omega_0)$, often called a realization, whereas if we fix the time $t = t_1$ it gives a random variable $x(t_1, \cdot)$ with a certain distribution, see Fig. 3.6.

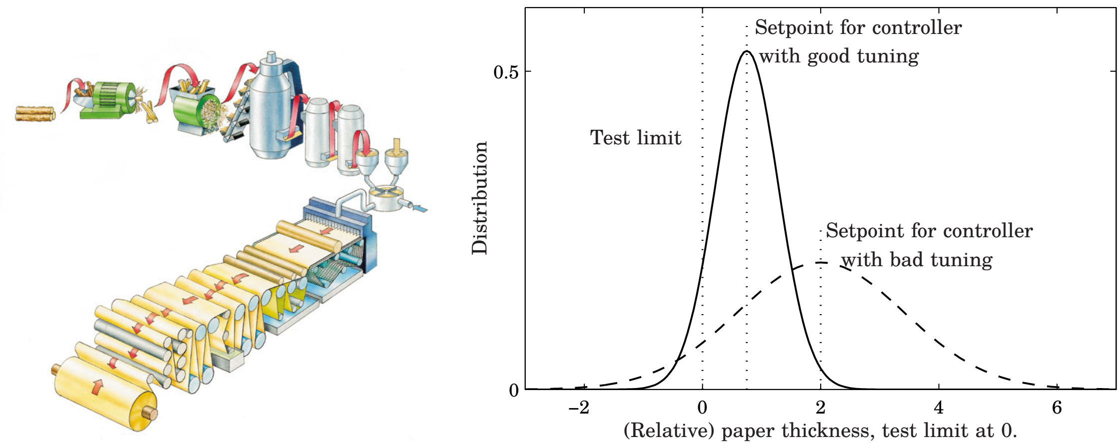

Figure 3.5 To be of acceptable quality, products must exceed a certain threshold. By minimizing the variance of the thickness we see that the average of the paper thickness can be reduced significantly (we come closer to the test limit) for the same yield. This may save a lot in production costs regarding both energy and raw material.

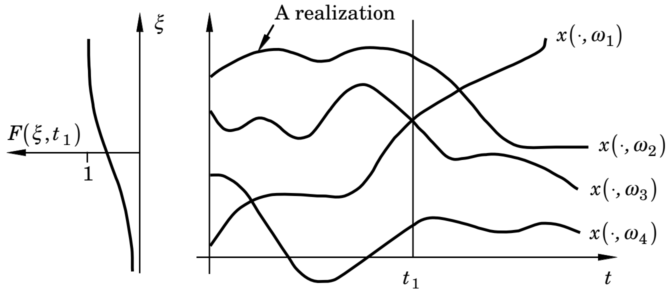

Figure 3.6 A stochastic process: For a fixed ω we call it a realization, for fixed time t_1 it will correspond to a random variable with a distribution $F(\xi, t_1) = Prob\{x(t_1) \leq \xi\}$.

For a zero-mean *stationary* stochastic processes the distribution is independent of t. We refer to the basic course in statistics for more details on the following concepts:

Mean-value function

$$\mathrm{E}\, x(t) \equiv 0$$

Covariance function. A zero mean Gaussian process x is completely determined by its covariance function:

$$R_x(\tau) = \mathrm{E}\, x(t+\tau)x(t)^T$$

Cross-covariance function

$$R_{xy}(\tau) = \mathrm{E}\, x(t+\tau)y(t)^T$$

Spectral density (defined for (weakly) stationary processes). The spectral density is the Fourier transform of the covariance function

$$\Phi_{xy}(\omega) = \int_{-\infty}^{\infty} R_{xy}(t)e^{-it\omega}\, dt$$

and

$$R_{xy}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega}\Phi_{xy}(\omega)\, d\omega$$

In particular, we get the following expressions for the *stationary covariance*:

$$\mathrm{E}\, xx^T = R_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega$$

When x is scalar, this is simply the variance of x. (Notation: We will use Φ_y as short for Φ_{yy}.)

For relations between covariance function, spectral density and a typical realization, see Fig. 3.7, where one may notice that the realizations seem to be "more random" the flatter the spectra is (over a larger frequency range) while peaks in the spectral density corresponds to periodic covariance functions.

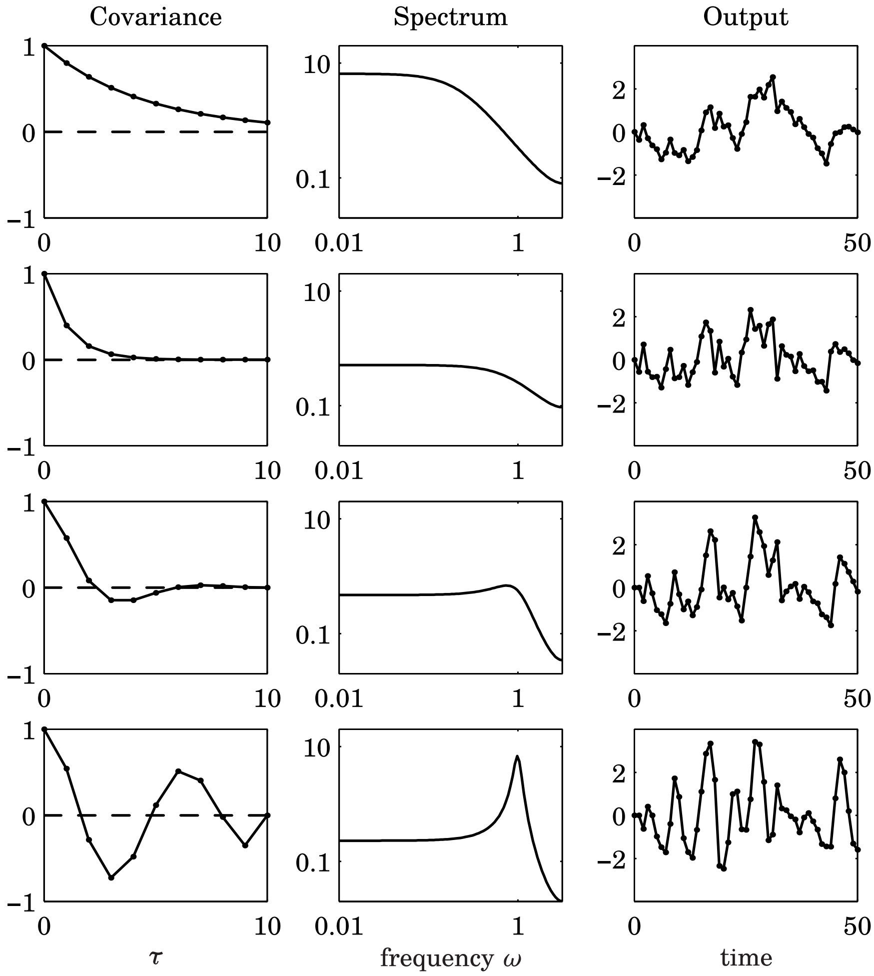

Figure 3.7 Relations between covariance function, spectral density and a typical realization. (Correction: The spectra should be divided by 2π)

White noise

A particular disturbance is so-called *white noise* w with *intensity* R_w. Here R_w is a constant matrix, which corresponds to a constant spectrum, totally flat and equal for all frequencies:

$$\Phi_w(\omega) = R_w$$

One effect of this definition is that the continuous-time version of white noise has infinite energy, and causes some issues to be handled mathematically rigorously, but we will not go into these details here.

The most important property of white noise which we will use later in the course, is that it can not be predicted; based on previous measurements there is no information about future values. From transform theory we also have that the Fourier transform of the Dirac pulse $\delta(t)$, is constant, which corresponds to an alternative interpretation: by applying a Dirac impulse as input to a linear system, the spectral density of the corresponding output (i.e., of the impulse response), will be like a finger-print of the system's frequency properties.

Two complementary problems related to modeling and analysis of stochastic disturbances y can now be formulated:

1. Determine the covariance function and spectral density of y when a stochastic process u is filered through a linear system,

$$Y(s) = G(s)U(s) \qquad \text{or} \qquad \begin{aligned} \dot{x} &= Ax + u \\ y &= Cx \end{aligned}$$

2. Conversely, find the parameters for a stable linear filter (transfer function $G(s)$ or state-space matrices A and C) to give the output y a desired spectral density.

These two problems will be studied in the remainder of this lecture.

Filtering of stochastic processes

For the first problem, we start with the *transfer function* representation

$$Y(i\omega) = G(i\omega)U(i\omega)$$

where $Y = \mathcal{F}\{y\}$, $U = \mathcal{F}\{u\}$ are the Fourier transforms. According to the definition, we get

$$\Phi_y(\omega) \hat{=} \Phi_{yy}(\omega) = Y(i\omega)Y(i\omega)^* = G(i\omega)U(i\omega)U(i\omega)^* G(i\omega)^*$$

where we can identify the spectral density of the output as

$$\Phi_{yy}(\omega) = G(i\omega)\Phi_{uu}(\omega)G(i\omega)^*$$

In similar way we find the cross-spectral density

$$\Phi_{yu}(\omega) = G(i\omega)\Phi_{uu}(\omega)$$

For a *state-space model*,

$$\dot{x} = Ax + Bw, \qquad \Phi_w(\omega) = R_w$$

we can calculate the transfer function from noise to state as

$$G_{xw}(s) = (sI - A)^{-1}B$$

and the spectral density for x will thus be

$$\Phi_x(\omega) = (i\omega I - A)^{-1}BR_w \underbrace{B^*(-i\omega I - A)^{-T}}_{((i\omega I - A)^{-1}B)^*}$$

The covariance matrix for state x is then given by

$$\Pi_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega)d\omega$$

However there is an alternative way of calculating Π_x:

Theorem 3.1
If all eigenvalues of A are in the left half-plane (i.e. $\mathrm{Re}\{\lambda_k\} < 0$) then there exists a unique matrix $\Pi_x = \Pi_x^T > 0$ which is the solution to the Lyapunov equation

$$A\Pi_x + \Pi_x A^T + BR_w B^T = 0$$

□

Example 2
Consider the system

$$\dot{x} = Ax + Bw \quad = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w$$

where w is scalar white noise with intensity $R_w = 1$. What is the stationary covariance of x?

First check the eigenvalues of A : $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \in LHP$. OK!

Then solve the Lyapunov equation

$$A\Pi_x + \Pi_x A^T + BR_w B^T = 0_{2\times 2}$$

$$\begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} + \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} [1 \quad 0] =$$
$$= \begin{bmatrix} 2(-\Pi_{11} + 2\Pi_{12} + 1 & -\Pi_{12} + 2\Pi_{22} - \Pi_{11} \\ -\Pi_{12} + 2\Pi_{22} - \Pi_{11} & -2\Pi_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving for Π_{11}, Π_{12} and Π_{22} gives

$$\Longrightarrow \Pi_x = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} > 0$$

In Matlab: `lyap([-1 2; -1 0],[1; 0]*[1 0])`

□

Spectral factorization

The next question is to go “backwards”, i.e., to figure out what filter can generate a certain spectrum.

- Assume that the signal y has spectrum $\Phi_y(\omega)$
- *(Spectral factorization)* Assume that we can find a transfer function $G(s)$ such that $G(i\omega)R_w G(i\omega)^* = \Phi_y(\omega)$ for a constant R_w.

In that case we can consider y as an output from the linear system G with white noise as input, $\Phi_w(\omega) = R_w$ (equal energy for all frequencies/flat spectrum).

THEOREM 3.2—SPECTRAL FACTORIZATION
Assume that the spectral density function $\Phi_y(\omega) \geq 0$ is a rational function of ω^2. Then there exists a rational function $G(s)$ with all poles strictly in the left half-plane and all zeros in the left half-plane or on the imaginary axis such that

$$\Phi_y(\omega) = |G(i\omega)|^2 = G(i\omega)G(-i\omega)$$

$\square$

If w and y are *scalar valued* and $\Phi_y(\omega)$ is a rational function of ω^2 it is easy to factorize to first or second order polynomials of ω^2 in both the numerator and the denominator. These can then be split in stable and unstable poles, respectively, and comes from the fact that if the characteristic polynomial for $G(i\omega)$ is $\Pi_{k=1}^{n}(i\omega - \lambda_k)$ then $G^* = G(-i\omega)$ will have its poles mirrored in the the imaginary axis.