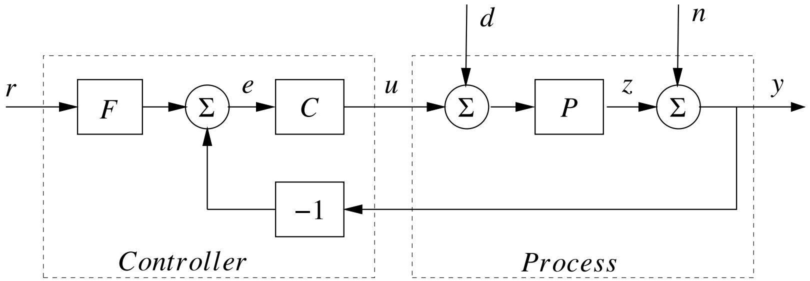
Lecture 3 – Outline

1. Control system specifications
2. Disturbance models
   - Stochastic processes
   - Filtering of white noise
   - Spectral factorization

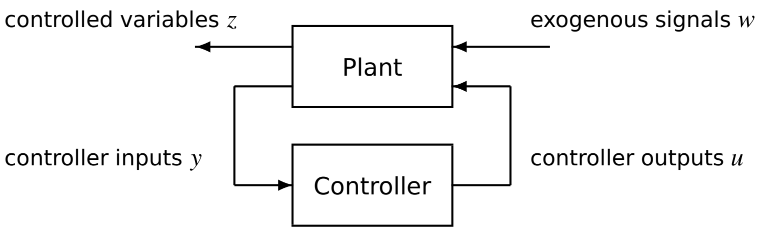
A basic control system



- Controller: feedback C, feedforward F
- Process: transfer function P
- Process/load disturbance d: drives system from desired state
- Controlled process variable z: should follow reference r
- Measurement noise n: corrupts information about z

A more general setting

Process disturbances need not enter at the process input, and measurement noise and setpoint values may also enter in different ways. More general setting:



We will return to this setting later in the course

Design specifications

Find a controller that

A: reduces the effect of load disturbances

B: does not inject too much measurement noise into the system

C: makes the closed loop insensitive to process variations

D: makes the output follow the setpoint

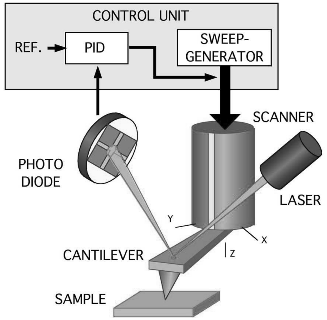
Common to have a controller with **two degrees of freedom** (2 DOF), i.e. separate signal transmission from y to u and from r to u. This gives a nice separation of the design problem:

1. Design feedback to deal with A, B, and C
2. Design feedforward to deal with D

Some systems only allow error feedback

Disk drive

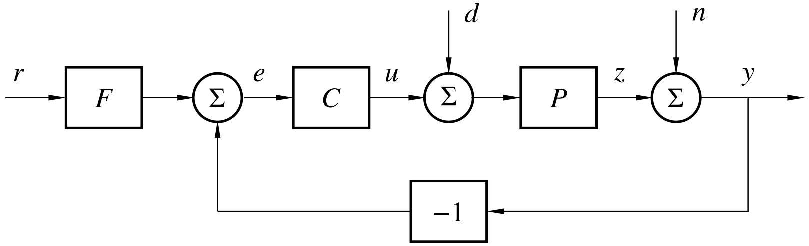
Atomic Force Microscope



Only the control error can be measured

Design of disturbance attenuation and setpoint response cannot be separated

Relations between signals



$$Z = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$
$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$
$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

The “Gang of Four” / “Gang of Six”

Four transfer functions are needed to characterize the response to load disturbances and measurement noise:

$$\frac{PC}{1+PC} \qquad \frac{P}{1+PC}$$
$$\frac{C}{1+PC} \qquad \frac{1}{1+PC}$$

Two more are required to describe the response to setpoint changes (for 2-DOF controllers):

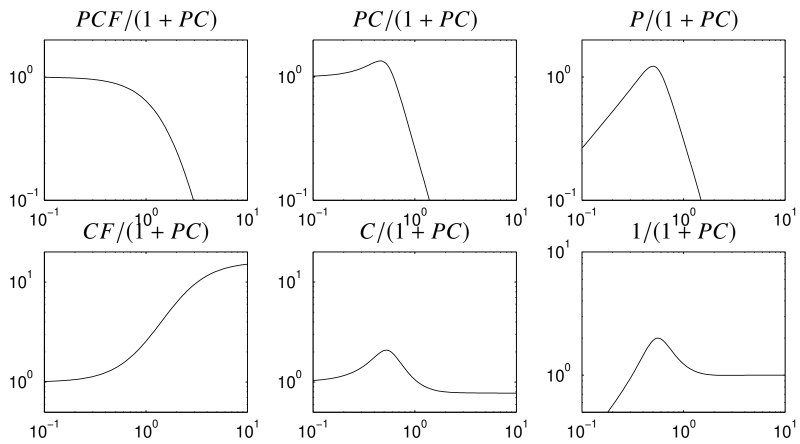
$$\frac{PCF}{1+PC} \qquad \frac{CF}{1+PC}$$

Some observations

- To fully understand a control system it is necessary to look at **all** four or six transfer functions
- It may be strongly misleading to show properties of only one or a few transfer functions, for example only the response of the output to command signals. (This is a common error.)
- The properties of the different transfer functions can be illustrated by their frequency or time responses.

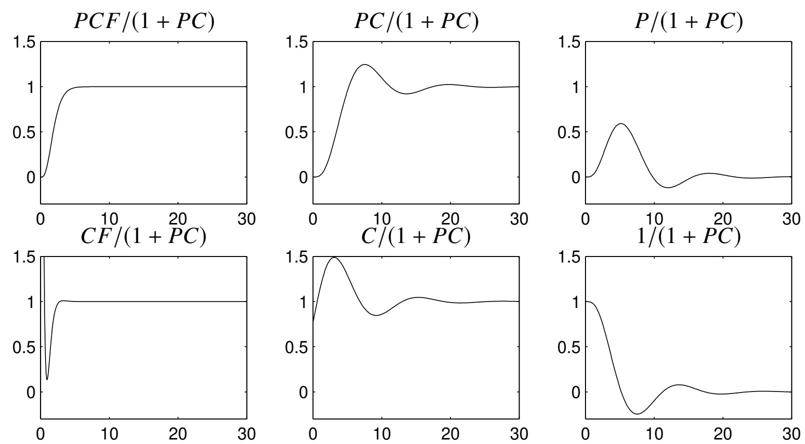
Example: Frequency Responses

PI control ($K_p = 0.775$, $T_i = 2.05$) of $P(s) = (s+1)^{-4}$ with $G_{yr}(s) = (0.5s+1)^{-4}$. Gain curves:



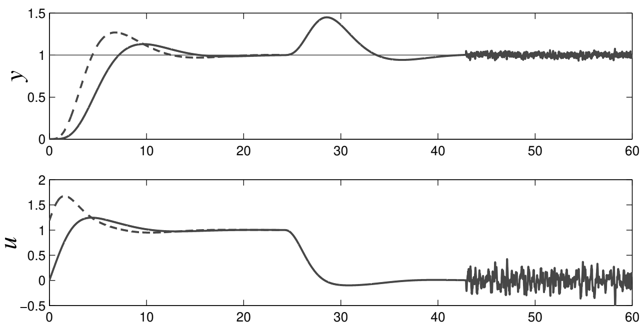
Example: Time Responses

PI control ($K_p = 0.775$, $T_i = 2.05$) of $P(s) = (s+1)^{-4}$ with $G_{yr}(s) = (0.5s+1)^{-4}$. Step responses:



Time responses—an alternative

Responses to setpoint step, load disturbance step and random measurement noise:

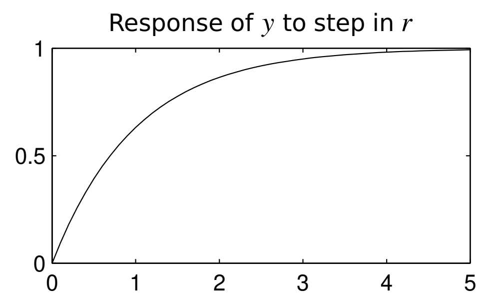


Error feedback (dashed), 2-DOF controller (full)

One plot gives a good overview!

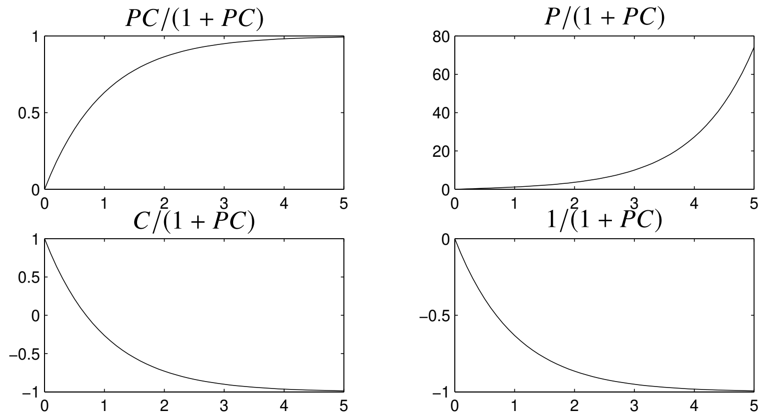
A warning

Remember to always look at **all** responses when you are dealing with control systems. The step response below looks fine, but. . .



A warning – Gang of Four

Step responses:



Unstable output response to load disturbance. What is going on?

A warning – The system

Process: $P(s) = \dfrac{1}{s-1}$

Controller: $C(s) = \dfrac{s-1}{s}$ (cancels the unstable process pole!)

Response to reference change:

$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{1}{s+1}$$

Reference to load disturbance:

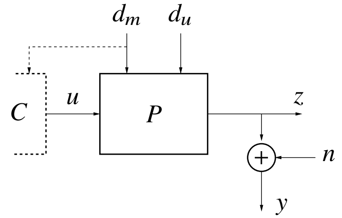
$$G_{yd}(s) = \frac{P}{1+PC} = \frac{s}{s^2-1} = \frac{s}{(s+1)(s-1)}$$

The control system is not internally stable!

Lecture 3 – Outline

1. Control system specifications
2. Disturbance models
   - Stochastic processes
   - Filtering of white noise
   - Spectral factorization

Two main types of disturbances



Process (or load) disturbances d

- Disturbances that affect the controlled process variables z
  - d_m measurable, can use feedforward to cancel them
  - d_n unmeasurable, must use feedback. Controller should have **high gain** at the dominant frequencies to supress them

Measurement disturbances n

- Disturbances that corrupt the feedback signals
  - Controller should have **low gain** at the dominant frequencies to avoid being “fooled”

Disturbance models

Deterministic disturbance models, e.g., impulse, step, ramp, sinusoidal signals

- Can be modeled by Dirac impulse filtered through linear system

Stochastic disturbance models

- Common model: Gaussian stochastic process
  - Can be modeled by white noise filtered through linear system
  - Reasonable model for many real-world random fluctuations

Mini-problem

What linear systems $G(s)$ can generate the following deterministic disturbances?

- A step

- A ramp

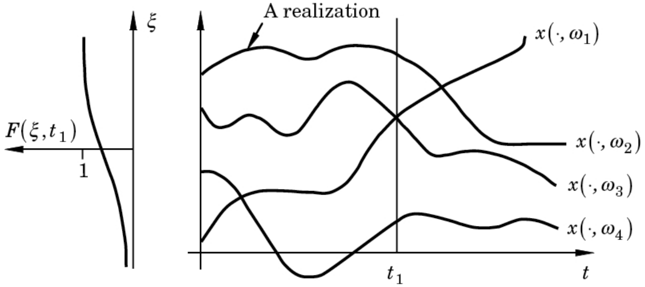
- A sinusoidal

Stochastic process – definition

A **stochastic process** is a family of random variables $\{x(t), t \in T\}$

Can be viewed as a function of two variables, $x = x(t, \omega)$:

- Fixed $\omega = \omega_0$ gives a time function $x(\cdot, \omega_0)$ (realization)
- Fixed $t = t_1$ gives a random variable $x(t_1, \cdot)$ (distribution)



For a **Gaussian process**, $x(t_1, \cdot)$ has a normal distribution

Gaussian processes

We will mainly work with **zero-mean stationary Gaussian processes**.

Mean-value function:

$$m_x = \mathrm{E}\, x(t) \equiv 0$$

Covariance function:

$$r_x(\tau) = \mathrm{E}\, x(t+\tau)x(t)^T$$

Cross-covariance function:

$$r_{xy}(\tau) = \mathrm{E}\, x(t+\tau)y(t)^T$$

A zero-mean stationary Gaussian process is completely characterized by its covariance function.

Spectral density

The **spectral density** or **spectrum** of a stationary stochastic process is defined as the Fourier transform of the covariance function:

$$\Phi_x(\omega) := \int_{-\infty}^{\infty} r_x(t) e^{-i\omega t}\, dt$$

- Describes the distribution of power over different frequencies

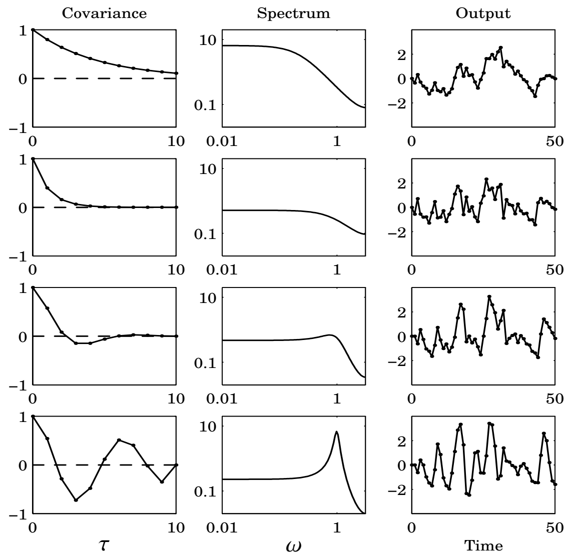
By inverse Fourier transform

$$r_x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}\Phi_x(\omega)\, d\omega$$

In particular, the **stationary (co)variance** is given by

$$\mathrm{E}\, x(t)x^T(t) = r_x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega)\, d\omega$$

Covariance fcn, spectral density, and realization



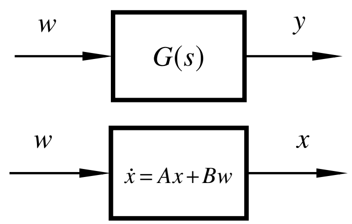
White noise

White noise with intensity R_w is a random process w with constant spectrum

$$\Phi_w(\omega) = R_w$$

- Variance is infinite – not physically realizable
- Can be interpreted as a train of random Dirac impulses
- When filtered through a stable LTI system, the output is a zero-mean stationary Gaussian process

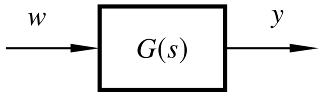
Filtering of white noise



Assume w white noise with intensity R_w. Two modeling/analysis problems:

1. Given $G(s)$ (or (A, B, C, D)), calculate the spectral density or stationary variance of y (or x)
2. Conversely, given the spectral density of y, determine a stable $G(s)$ that generates that spectrum
   - Known as **spectral factorization**

Calculation of spectrum – transfer function form



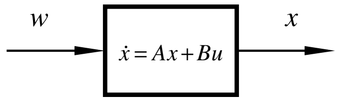
Given stable $G(s)$ and input w with the spectral density $\Phi_w(\omega)$.
Then output y gets the spectrum

$$\Phi_y(\omega) = G(i\omega)\Phi_w(\omega)G^*(i\omega)$$

Special case: If w is white noise with intensity R_w, then

$$\Phi_y(\omega) = G(i\omega)R_w G^*(i\omega)$$

Calculation of spectrum – state-space form



Assume a stable linear system with white noise input

$$\dot{x} = Ax + Bw, \qquad \Phi_w(\omega) = R_w$$

The transfer function from w to x is

$$G(s) = (sI - A)^{-1}B$$

and the spectrum for x will be

$$\Phi_x(\omega) = (i\omega I - A)^{-1}BR_w \underbrace{B^*(-i\omega I - A)^{-T}}_{G^*(i\omega)}$$

Calculation of stationary covariance – state-space form

Theorem 3.1

Given a stable linear system with white noise input

$$\dot{x} = Ax + Bw, \qquad \Phi_w(\omega) = R_w$$

then the stationary covariance of x is given by

$$\mathrm{E}\, xx^T = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega)d\omega := \Pi_x$$

where $\Pi_x = \Pi_x^T > 0$ is given by the solution to the Lyapunov equation

$$A\Pi_x + \Pi_x A^T + BR_w B^T = 0$$

Calculation of covariance – example

Consider the system

$$\dot{x} = Ax + Bw = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w$$

where w is white noise with intensity 1.

What is the stationary covariance of x?

First check the eigenvalues of $A : \lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \in LHP$. OK!

Solve the Lyapunov equation $A\Pi_x + \Pi_x A^T + BR_w B^T = 0_{2,2}$.

Example cont'd

$$A\Pi_x + \Pi_x A^T + BR_w B^T = 0_{2\times 2}$$

Find Π_x:

$$\begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} + \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2(-\Pi_{11} + 2\Pi_{12}) + 1 & -\Pi_{12} + 2\Pi_{22} - \Pi_{11} \\ -\Pi_{12} + 2\Pi_{22} - \Pi_{11} & -2\Pi_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving for Π_{11}, Π_{12} and Π_{22} gives

$$\Pi_x = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} > 0$$

Matlab: `lyap([-1 2; -1 0], [1; 0]*[1 0])`

Spectral factorization

Theorem 3.2

Assume that the scalar spectral density function $\Phi_w(\omega) \geq 0$ is a rational function of ω^2 and finite for all ω. Then there exists a rational function $G(s)$ with all poles in the left half-plane and all zeros in the left half-plane or on the imaginary axis such that

$$\Phi_w(\omega) = |G(i\omega)|^2 = G(i\omega)G(-i\omega)$$

Spectral factorization — example

Find a stable, minimum-phase filter $G(s)$ such that a process y generated by filtering unit intensity white noise through G gives

$$\Phi_y(\omega) = \frac{\omega^2 + 4}{\omega^4 + 10\omega^2 + 9},$$

Solution. We have

$$\Phi_y(\omega) = \frac{\omega^2 + 4}{(\omega^2 + 1)(\omega^2 + 9)} = \left|\frac{i\omega + 2}{(i\omega + 1)(i\omega + 3)}\right|^2$$

implying

$$G(s) = \frac{s + 2}{(s + 1)(s + 3)}$$

Lecture 3 – summary

- Look at all important closed-loop transfer functions: Gang of four / gang of six
- White noise filtered through LTI system gives Gaussian stochastic process – simple but useful disturbance model
- Calculation of spectrum and stationary covariance given generating system
- Calculation of generating system given spectrum (spectral factorization)