Lecture 2

Stability and Robustness*

This lecture discusses the role of stability in feedback design. The emphasis is not on yes/no tests for stability, but rather on how to measure the distance to instability. The small gain theorem is introduced as a means to verify stability in presence of model uncertainty.

2.1 Stability of feedback systems

Feedback and stability are closely connected issues. On one hand, the introduction of feedback may potentially create instability. On the other hand, properly applied feedback is often the best way to get rid of instabilities.

There are several well known examples in the history. One is the construction of airplanes. The first airplanes in the early 1900s were depending on the pilot to stabilize the dynamics manually using the control-stick. The modern fighter JAS-Gripen was built unstable for the sake of maneuverability and relies on computer control for stabilization.

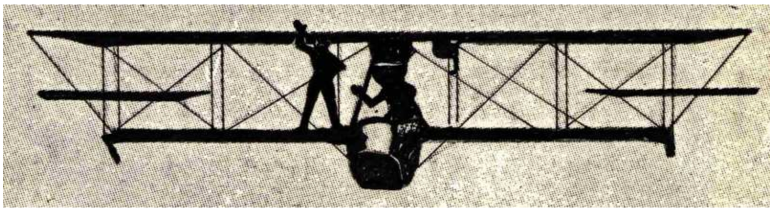


Figure 2.1 Lawrence Sperry demonstrates a stabilizing gyroscopic controller. He waves his hand in the air, while his mechanic is walking on the wing.

Another striking example was the construction of the first electronic feedback amplifiers, that were necessary to build long distance telephone connections in the 1930s. In this case, high gain feedback was needed to reduce the nonlinear signal distortion. Stability problems became a major issue, and the development of frequency domain stability criteria was critical for successful implementation.

A modern example is the maneuver test for Mercedes A-class, the so called *elk-test*, that created unstable oscillations severe enough to turn the car over. The problem was solved by introducing electronic feedback control.

The L_2 gain of a system was defined in the previous lecture, and we will call a system *input-output stable* (or *L_2 stable*) if its L_2 gain is bounded. For an LTI system $\mathcal{S}$ with impulse response $g(t)$ and transfer function $G(s)$, the following stability conditions are equivalent:

- $\|\mathcal{S}\|$ is bounded.
- $g(t)$ decays exponentially, i.e., there exist constants $\alpha, \beta > 0$ such that $|g(t)| < \alpha e^{-\beta t}$ for $t \geq 0$.

*Written by A. Rantzer with contributions by K.J. Åström

Figure 2.2 The stability problems of Mercedes A-class were solved by electronic feedback

- $\int_0^\infty |g(t)| dt$ is bounded.
- All poles of $G(s)$ are in the left half-plane (i.e., all poles have negative real part).

Example 1 Let us determine input-output stability of the following systems

$$(I) \qquad \begin{cases} \dot{x} = \begin{bmatrix} -1 & 2 \\ -3 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 \end{bmatrix} x + u \end{cases}$$

$$(II) \qquad y(t) = \int_0^t e^{\tau - t}\Big(u(\tau) - y(\tau)\Big) d\tau$$

(I) The first system is input-output stable due to stable eigenvalues of the system matrix. A two-by-two matrix like this is stable if and only if the trace is negative (here -3) and the determinant is positive (here 8). This is because the trace is the sum of the eigenvalues and the determinant is the product. In general, eigenvalues can be computed by the matlab command `eig(A)`:

```
>> eig([-1 2; -3 -2])

ans =
  -1.5000 + 2.3979i
  -1.5000 - 2.3979i
```

Note that the coefficients of the characteristic polynomial should not be computed, at least for high order systems, since this generally leads to numerical difficulties.

(II) The input-output relationship can be written

$$y = g * (u - y)$$

where $g(t) = e^{-t}$, $t \geq 0$. After Laplace transformation, this gives

$$\begin{aligned} & Y(s) = \frac{1}{s+1}[U(s) - Y(s)] \\ \Rightarrow \qquad & Y(s) = \frac{1}{s+2} U(s) \end{aligned}$$

The transfer function $1/(s+2)$ shows that the system is stable, since the only pole -2 is negative.

□

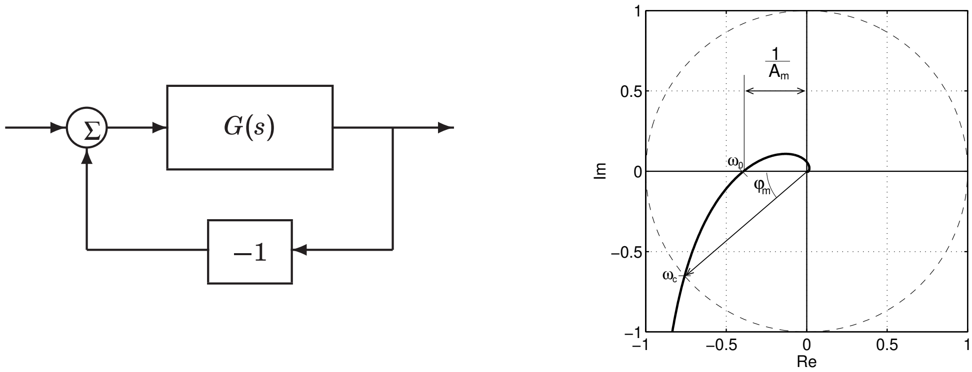


Figure 2.3 The closed-loop system remains stable as long as the Nyquist diagram does not encircle -1. The amplitude margin A_m and phase margin ϕ_m measure the distance from instability.

Our main objective is to study stability of feedback loops. From the basic course, we recall the Nyquist criterion, which supports understanding by graphical illustrations.

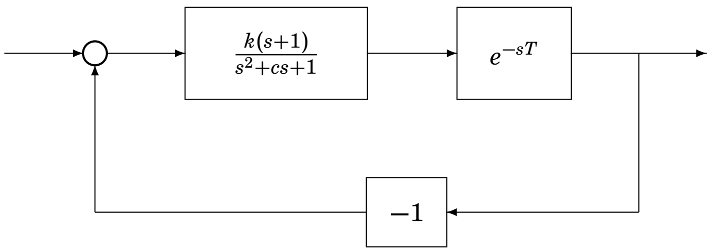
THEOREM 2.1—THE NYQUIST CRITERION
Suppose that $G(s)$ is stable and that the Nyquist plot $G(i\omega)$, $\omega \in \mathbf{R}$ does not encircle -1. Then $(1+G(s))^{-1}$ is also stable. □

The following example illustrates the use of the theorem.

Example 2 As motivating example, consider a position control in a mechanical system with damping coefficient c. The controller contains a time delay:

$$\ddot{x} + c\dot{x} + x = u \qquad\qquad u(t) = -k[x(t-T) + \dot{x}(t-T)]$$

The feedback loop is illustrated in the figure below. Nominal values of the parameters are $k = 1$, $c = 1$ and $T = 0$. Let us investigate how much margin there is in each of the parameters before the system becomes unstable?



For this purpose, we plot the Nyquist and Bode diagrams of the nominal transfer function $(s+1)/(s^2+s+1)$. The Matlab command `margin` gives numerical values for the amplitude- and phase-margins.

For $k = c = 1$ the open-loop transfer function is

$$\frac{s+1}{s^2+s+1}e^{-sT}$$

For small values of T the Nyquist plot will not encircle -1, so the systems remains stable. To find out exactly how large values of T are needed for instability, note that the phase margin 109 degrees (or $109\pi/180$ radians) is obtained at the frequency 1.41 rad/sec. Hence a time delay of $\frac{109\pi}{180\cdot 1.41} = 1.35$ seconds can be tolerated for $k = c = 1$.

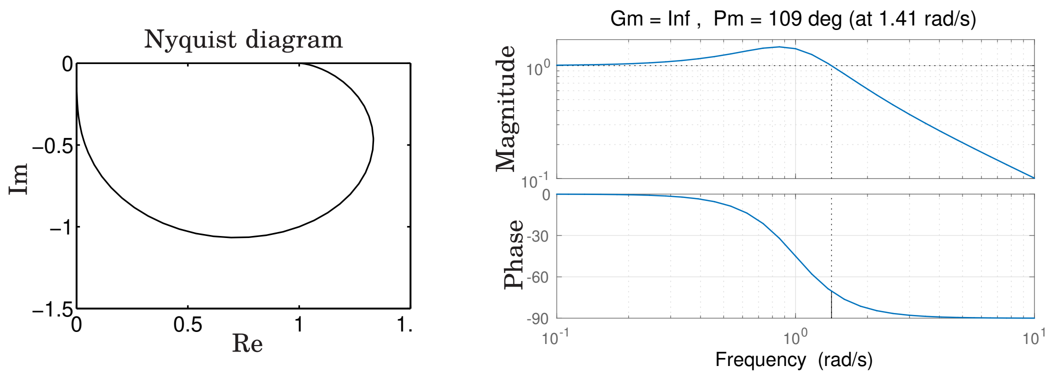


Figure 2.4 Nyquist and bode plots for the nominal transfer function $(s+1)/(s^2+s+1)$

To investigate the robustness to variations in the parameters c and k, we note that the closed-loop characteristic polynomial for $T = 0$ is

$$(s^2 + cs + 1) + k(s + 1) = s^2 + (c + k)s + 1 + k$$

The stability condition for a second degree polynomial is positive coefficients, so stability is maintained as long as $c + k > 0$ and $1 + k > 0$. $\square$

Having analyzed the example with respect to parametric uncertainty above, it is natural to ask about robustness to unmodelled dynamics. For example, this would be relevant to capture the difference between a rigid body and an elastic one. This is also the subject for the remaining part of the lecture.

2.2 Sensitivity

Two transfer functions are of particular interest in the study of the feedback loop below.

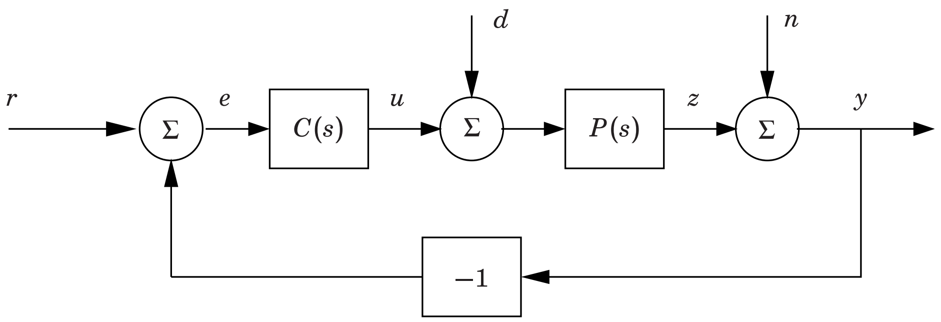


Figure 2.5 A simple control loop

These are

$$S(s) = \frac{1}{1 + P(s)C(s)} \qquad \text{(the } \textit{sensitivity function}\text{)}$$

$$T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)} \qquad \text{(the } \textit{complementary sensitivity function}\text{)}$$

The term complementary refers to the fact that $S(s) + T(s) \equiv 1$. Note that $T(s)$ is the transfer function from reference signal r to the the plant output z. Another name for $T(s)$ is therefore the closed-loop transfer function of the system, while $P(s)C(s)$ is called the open-loop transfer function or just the “loop transfer function”.

$$R^{-1} = \sup_{\omega} \left| \frac{1}{1 + C(i\omega)P(i\omega)} \right|$$

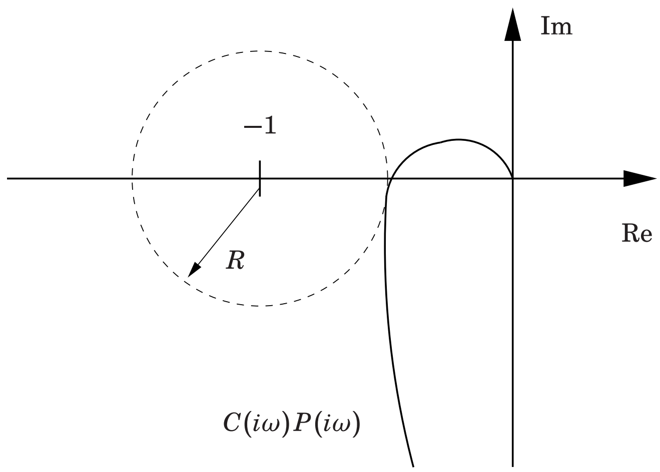


Figure 2.6 The L_2 gain of the sensitivity function is the inverse of the distance from the Nyquist plot to -1.

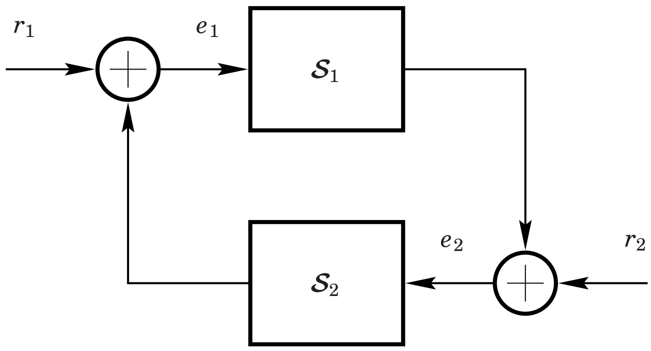
The name sensitivity function refers to the fact that S measures how small relative errors in P are mapped into relative errors in T. This is verified by a simple calculation:

$$\frac{dT}{dP} = \frac{d}{dP}\left(1 - \frac{1}{1+PC}\right) = \frac{C}{(1+PC)^2} = \frac{TS}{P} \qquad \Rightarrow \qquad \frac{dT/T}{dP/P} = S$$

Notice that T is a nonlinear function of P. Hence the sensitivity calculation only says something about the response to small pertubations in P. For larger perturbations, the system could have a drastically different behaviour and even become unstable.

Given the Nyquist criterion, it is natural to conjecture that the robustness to unmodelled dynamics should somehow be related to the distance from the Nyquist plot to the point -1. It is therefore striking to note that the L_2 gain of the sensitivity function turns out to be exactly the inverse of this distance. Consequences are investigated in the next section.

2.3 Robustness via the small gain theorem



In this section, we will investigate stability robustness using the following theorem, based on the notion of input-output L_2 gain. For simplicity, calculations are done assuming zero initial conditions.

Theorem 2.2—The small gain theorem
Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are input-output stable systems with L_2 gain $\|\mathcal{S}_1\|$ and $\|\mathcal{S}_2\|$. If $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$, then the L_2 gain from (r_1, r_2) to (e_1, e_2) in the closed-loop system

$$e_1 = \mathcal{S}_2(e_2) + r_1$$
$$e_2 = \mathcal{S}_1(e_1) + r_2$$

is finite. □

Proof. Define $\|y\|_T = \sqrt{\int_0^T |y(t)|^2 dt}$. Then $\|\mathcal{S}(y)\|_T \leq \|\mathcal{S}\| \cdot \|y\|_T$.

$$
\begin{aligned}
e_1 &= r_1 + \mathcal{S}_2(r_2 + \mathcal{S}_1(e_1)) \\
\|e_1\|_T &\leq \|r_1\|_T + \|\mathcal{S}_2\|\Big(\|r_2\|_T + \|\mathcal{S}_1\| \cdot \|e_1\|_T\Big) \\
\|e_1\|_T &\leq \frac{\|r_1\|_T + \|\mathcal{S}_2\| \cdot \|r_2\|_T}{1 - \|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\|}
\end{aligned}
$$

Bounded gain from (r_1, r_2) to e_1 follows as $T \to \infty$. The gain to e_2 is bounded in the same way. □

To demonstrate how the small gain theorem can be used for robustness analysis, consider the feedback loop in Figure 2.7, where the plant $P(s)$ has been replaced by $[1 + \Delta(s)]P(s)$.

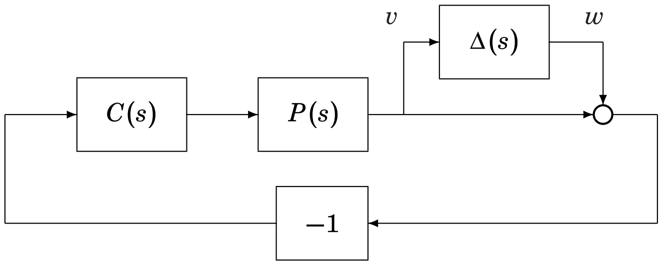


Figure 2.7 Loop diagram with perturbed plant $[1 + \Delta(s)]P(s)$

The transfer function from w to v is equal to $-T(s)$, the complementary sensitivity function. Hence, by the small gain theorem, the feedback system remains stable as long as

$$\|\Delta\| \cdot \|T\| < 1$$

Note that the small gain theorem does not assume linearity or time-invariance. Hence the closed-loop system will remain stable for all plants of the form $P(s)[1 + \Delta(s)]$ where Δ has L_2 gain smaller than $[\sup_\omega |T(i\omega)|]^{-1}$, even for Δ that are nonlinear or time-varying.

As a second example, let us derive a stability criterion for the case that the perturbation appears additively, i.e. $P(s)$ is replaced by $P(s) + \Delta(s)$.

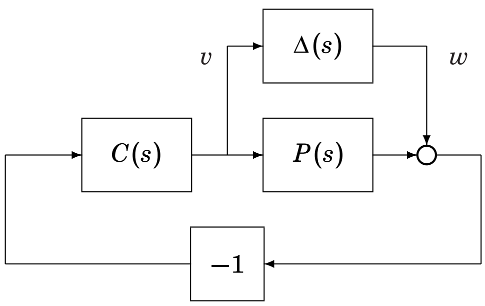


Figure 2.8 Loop diagram with perturbed plant $P(s) + \Delta(s)$

Then the transfer function from w to v is equal to $C(s)S(s)$, so the small gain theorem shows stability for all perturbations satisfying

$$\|\Delta\| \cdot \|CS\| < 1$$

2.4 Gain of multivariable systems

In the previous chapter we looked at norms of signals and gains of systems. Recall that the L_2 gain of a SISO system was the largest magnitude in the Bode diagram. Later in the course we will work with multivariable systems where the relation between vectors of inputs and outputs could be described as a matrix with transfer function elements, as for example

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{4}{2s+1} \\ \dfrac{s}{s^2+s+1} & \dfrac{3}{s+4} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{2.1}$$

For that purpose we will first introduce how to measure the size of a vector (vector norm) and based on this the induced norm (gain) of a matrix.

For a vector $x \in \mathbb{C}^n$, we use the L_2 norm

$$|x| = \sqrt{x^* x} = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

where * denotes the conjugate transpose.

For matrices $A \in \mathbb{C}^{n\times m}$, we use the L_2-induced norm, the largest ratio between the output and the corresponding input

$$\|A\| := \sup_x \frac{|Ax|}{|x|} = \sup_x \sqrt{\frac{x^* A^* A x}{x^* x}} = \sqrt{\overline{\lambda}(A^* A)}$$

Here $\overline{\lambda}(A^* A)$ denotes the largest eigenvalue of $A^* A$. The largest gain is thus $\sqrt{\overline{\lambda}(A^* A)}$, also called the *largest singular value* of A , denoted $\overline{\sigma}(A)$. The fraction $|Ax|/|x|$ is maximized when x is a corresponding eigenvector to $A^* A$, but note that we in general have different directions of the input vector x and the output vector Ax (as x in general is not a eigenvector of A).

The following illustrates how to use singular value decomposition to calculate the gain of the static system

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Matlab-code for singular value decomposition of the matrix

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

Singular Value Decomposition (SVD) :

$$A = U \cdot S \cdot V^*$$

where both the matrices U and V are unitary (i.e. have orthonormal columns s.t. $V^* \cdot V = I$) and S is the diagonal matrix with (sorted decreasing) singular values σ_i.
Multiplying A with an input vector along the first column in V gives

$$A \cdot V_{(:,1)} = USV^* \cdot V_{(:,1)} = $$
$$= US \begin{bmatrix} 1 \\ 0 \end{bmatrix} = U_{(:,1)} \cdot \sigma_1$$

That is, we get maximal gain σ_1 in the output direction $U_{(:,1)}$ if we use an input in direction $V_{(:,1)}$ (and minimal gain $\sigma_n = \sigma_2$ if we use the last column $V_{(:,n)} = V_{(:,2)}$).

```
>> A=[2 4 ; 0 3]
A =
     2     4
     0     3
>> [U,S,V]=svd(A)
U =
    0.8416   -0.5401
    0.5401    0.8416
S =
    5.2631         0
         0    1.1400
V =
    0.3198   -0.9475
    0.9475    0.3198

>> A*V(:,1)
ans =
    4.4296
    2.8424

>> U(:,1)*S(1,1)
ans =
    4.4296
    2.8424
```

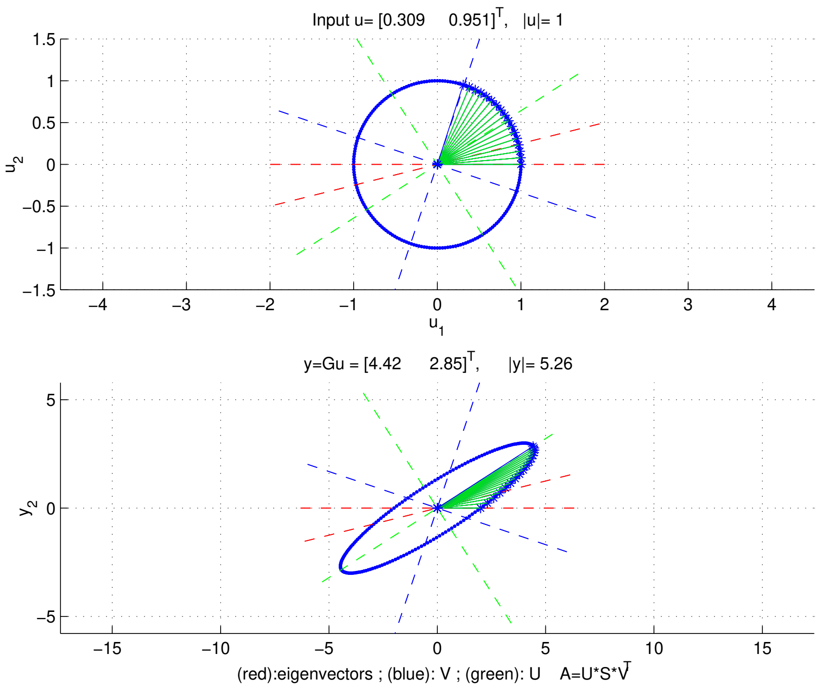


Figure 2.9 Matlab-example showing how the output vector $y = Au$ changes size and direction when the input signal with $||u|| = 1$ are changed in different directions around the unit circle. (Note scaling of axes in lower plot).

Now when we know how to calculate the induced norm for a matrix, we can apply this to a transfer function matrix, where the elements depend on the frequency. The magnitude plot of the Bode diagram for single-input-single-output systems has the multivariable correspondence of plotting all singular values for the matrix transfer function as a function of the frequency. In Matlab this is done with the command `sigma`, see example below.

The *gain* of the system G is defined as

$$\|G\|_\infty = \max_\omega |G(i\omega)|$$

and is the largest magnitude (largest singular value) when sweeping over all frequencies for $|G(i\omega)|$, see Fig 2.10.

Example: Consider the transfer matrix $G(s)$ from Eq. (2.1)

$$G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{4}{2s+1} \\ \dfrac{s}{s^2+0.1s+1} & \dfrac{3}{s+1} \end{bmatrix}$$

In Matlab we can plot the singular values over all frequencies and calculate the system gain as follows:

```
>> s=tf('s')
>> G=[ 2/(s+1) 4/(2*s+1); s/(s^2+0.1*s+1) 3/(s+1)];
>> sigma(G) % plot sigma values of G  wrt fq
>> grid on
```

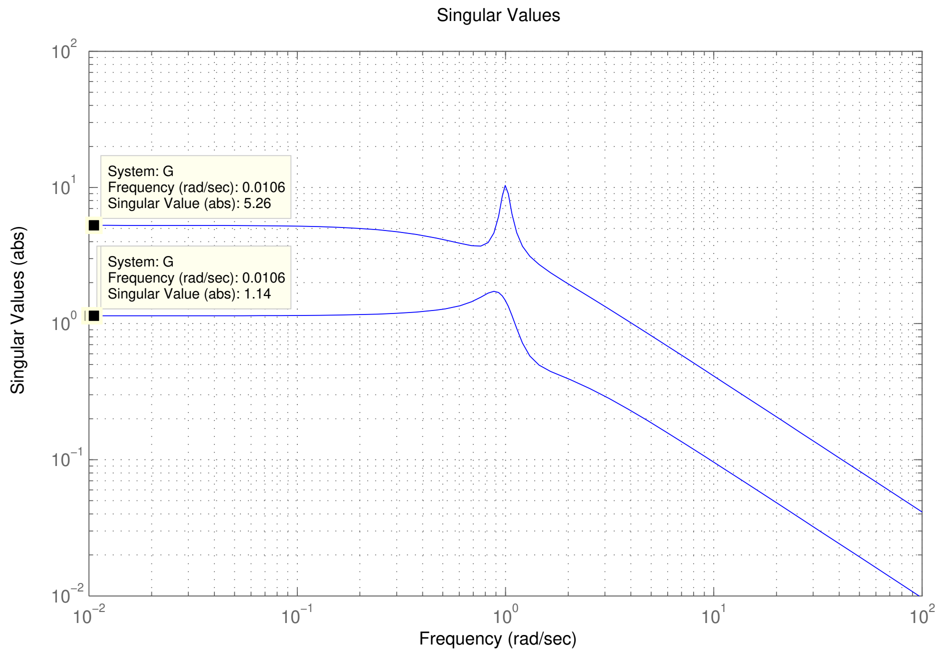


Figure 2.10 The singular values of the tranfer function matrix in Eq. (2.1). Note that G(0)=[2,4 ; 0 3] which corresponds to M in the SVD-example above. $\|G\|_\infty = 10.3577$.

```
>> norm(G,inf)  % infinity norm = system gain
   ans =
      10.3577
```