Lecture 2 – Outline

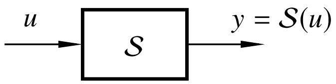
1. Stability
2. Sensitivity and robustness
3. The Small Gain Theorem
4. Singular values

Stability is crucial

Examples:

- bicycle
- JAS 39 Gripen
- Mercedes A-class
- ABS brakes

Input–output stability



A general system $\mathcal{S}$ is called **input–output stable** (or "L_2 stable" or "BIBO stable" or just "stable") if its L_2 gain is finite:

$$\|\mathcal{S}\| = \sup_u \frac{\|\mathcal{S}(u)\|}{\|u\|} < \infty$$

Input–output stability of LTI systems

For an LTI system $\mathcal{S}$ with impulse response $g(t)$ and transfer function $G(s)$, the following stability conditions are equivalent:

- $\|\mathcal{S}\|$ is bounded
- $g(t)$ decays exponentially
- All poles of $G(s)$ are in the left half-plane (LHP), i.e., all poles have negative real part

Internal stability

The LTI system

$$\frac{dx}{dt} = Ax + Bu$$
$$y = Cx + Du$$

is called **internally stable** if the following equivalent conditions hold:

- The state x decays exponentially when $u = 0$
- All eigenvalues of A are in the LHP

Internal vs input–output stability

If

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

is internally stable **then**

$$G(s) = C(sI - A)^{-1}B + D$$

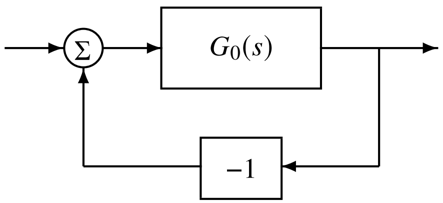
is input–output stable.

Warning

The opposite is not always true! There may be unstable pole-zero cancellations (that also render the system uncontrollable and/or unobservable), and these may not be seen in the transfer function!

Stability of feedback loops

Assume scalar open-loop system $G_0(s)$



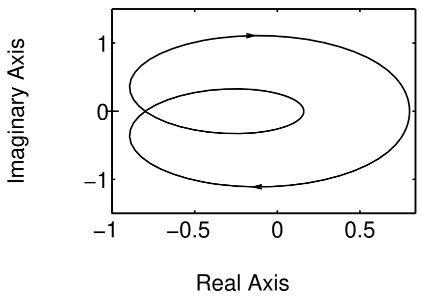
The closed-loop system is stable **if and only if** all solutions to the characteristic equation

$$1 + G_0(s) = 0$$

are in the left half-plane.

Simplified Nyquist criterion

If $G_0(s)$ is stable, then the closed-loop system $[1 + G_0(s)]^{-1}$ is stable **if and only if** the Nyquist curve of $G_0(s)$ does not encircle -1.



(Note: Matlab gives a Nyquist plot for both positive and negative frequencies)

General Nyquist criterion

Let

- P = number of **unstable** (RHP) poles in $G_0(s)$
- N = number of **clockwise** encirclements of -1 by the Nyquist plot of $G_0(s)$

Then the closed-loop system $[1 + G_0(s)]^{-1}$ has $P + N$ unstable poles

Sensitivity and robustness

- How sensitive is the closed-loop system to model errors and disturbances?
- How do we measure the “distance to instability”?
- Is it possible to guarantee stability for all systems within some distance from the ideal model?

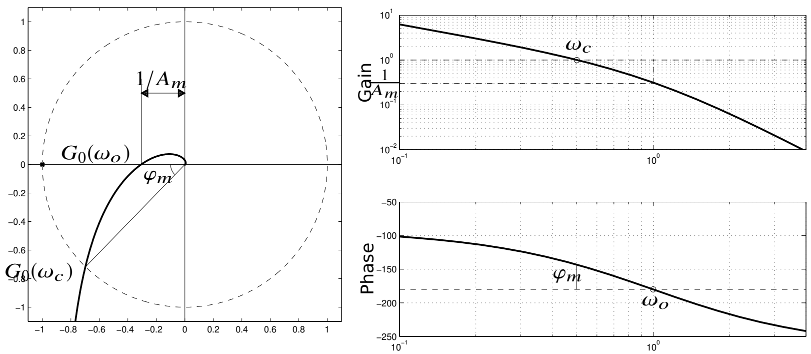
Amplitude and phase margins

Amplitude margin A_m:

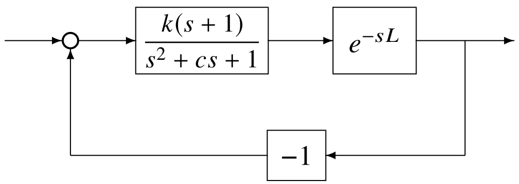
$$\arg G_0(i\omega_0) = -180^\circ, \qquad |G_0(i\omega_0)| = 1/A_m$$

Phase margin φ_m:

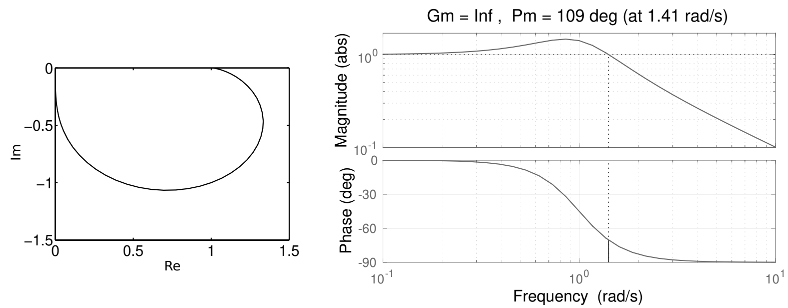
$$|G_0(i\omega_c)| = 1, \qquad \arg G_0(i\omega_c) = \varphi_m - 180^\circ$$



Mini-problem

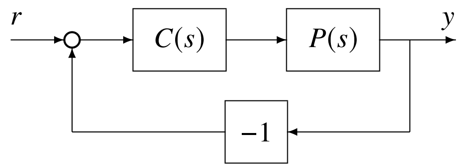


Nominally $k = 1$, $c = 1$ and $L = 0$. How much margin is there in each parameter before the closed-loop system becomes unstable?



Mini-problem

Sensitivity functions



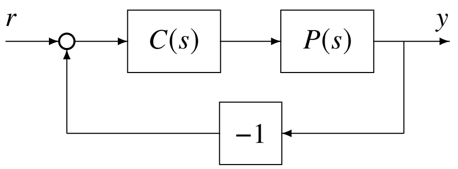
$$S(s) = \frac{1}{1 + P(s)C(s)}$$ sensitivity function

$$T(s) = \frac{P(s)C(s)}{1 + P(s)C(s)}$$ complementary sensitivity function

Note that we always have

$$S(s) + T(s) = 1$$

Sensitivity towards changes in plant



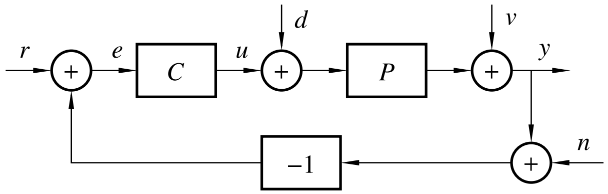
How sensitive is the closed loop to a (small) change in P?

$$\frac{dT}{dP} = \frac{C}{(1 + PC)^2} = \frac{T}{P(1 + PC)}$$

Relative change in T compared to relative change in P:

$$\frac{dT/T}{dP/P} = \frac{1}{1 + PC} = S$$

Sensitivity towards disturbances



Open-loop response ($C = 0$) to process disturbances d, v:

$$Y_{ol} = V + PD$$

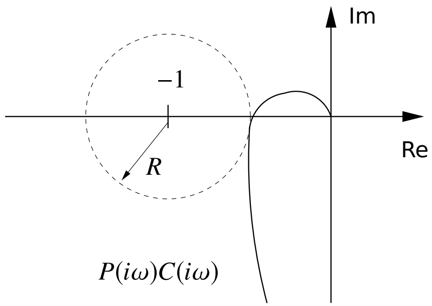
Closed-loop response:

$$Y_{cl} = \frac{1}{1 + PC}V + \frac{P}{1 + PC}D = S\,Y_{ol}$$

Interpretation as stability margin

The L_2 gain of the sensitivity function measures the inverse of the distance between the Nyquist plot and the point -1:

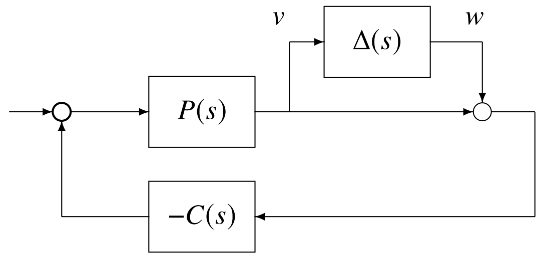
$$R^{-1} = \sup_{\omega} \left| \frac{1}{1 + P(i\omega)C(i\omega)} \right| = M_s$$



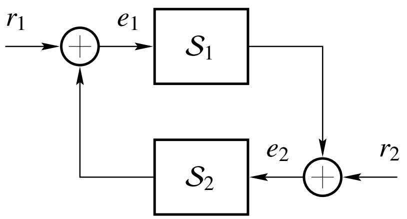
Robustness analysis

How large plant uncertainty Δ can be tolerated without risking instability?

Example (multiplicative uncertainty):



The Small Gain Theorem



Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are stable. **If** $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$, **then** the closed-loop system (from (r_1, r_2) to (e_1, e_2)) is stable.

- Note 1: The theorem applies also to nonlinear, time-varying, and multivariable systems
- Note 2: The stability condition is sufficient but not necessary, so the results may be conservative

Proof

$$e_1 = r_1 + \mathcal{S}_2(r_2 + \mathcal{S}_1(e_1))$$

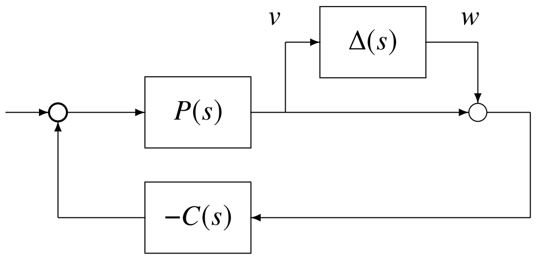
$$\|e_1\| \leq \|r_1\| + \|\mathcal{S}_2\| \Big(\|r_2\| + \|\mathcal{S}_1\| \cdot \|e_1\| \Big)$$

$$\|e_1\| \leq \frac{\|r_1\| + \|\mathcal{S}_2\| \cdot \|r_2\|}{1 - \|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\|}$$

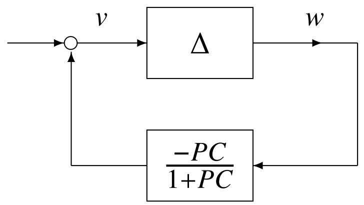
This shows bounded gain from (r_1, r_2) to e_1.

The gain to e_2 is bounded in the same way.

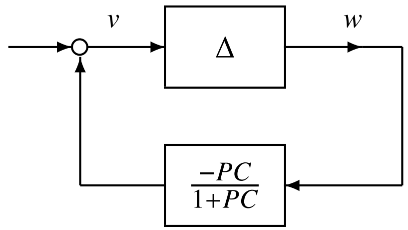
Application to robustness analysis



The diagram can be redrawn as



Application to robustness analysis



Assuming that $T = \frac{PC}{1+PC}$ is stable, The Small Gain Theorem guarantees stability if

$$\|\Delta\| \cdot \|T\| < 1$$

Gain of multivariable systems

Recall from Lecture 1 that

$$\|\mathcal{S}\| = \sup_{\omega} |G(i\omega)| = \|G\|_\infty$$

for a stable LTI system $\mathcal{S}$.

How to calculate $|G(i\omega)|$ for a multivariable system?

Vector norm and matrix gain

For a vector $x \in \mathbf{C}^n$, we use the 2-norm

$$|x| = \sqrt{x^* x} = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

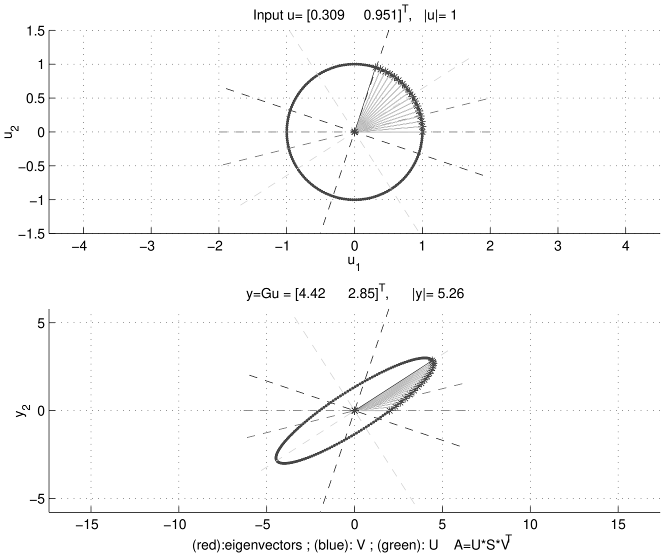
(A^* denotes the conjugate transpose of A)

For a matrix $A \in \mathbf{C}^{n\times m}$, we use the L_2-induced norm

$$\|A\| := \sup_x \frac{|Ax|}{|x|} = \sup_x \sqrt{\frac{x^*A^*Ax}{x^*x}} = \sqrt{\overline{\lambda}(A^*A)}$$

$\overline{\lambda}(A^*A)$ denotes the largest eigenvalue of A^*A. The ratio $|Ax|/|x|$ is maximized when x is a corresponding eigenvector.

Example: Different gains in different directions: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



Singular Values

For a matrix A, its singular values σ_i are defined as

$$\sigma_i = \sqrt{\lambda_i}$$

where λ_i are the eigenvalues of A^*A.

Let $\overline{\sigma}(A)$ denote the largest singular value and $\underline{\sigma}(A)$ the smallest singular value.

For a linear map $y = Ax$, it holds that

$$\underline{\sigma}(A) \le \frac{|y|}{|x|} \le \overline{\sigma}(A)$$

The singular values are typically computed using singular value decomposition (SVD):

$$A = U\Sigma V^*$$

SVD example

Matlab code for singular value decomposition of the matrix

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

SVD:

$$A = U \cdot S \cdot V^*$$

where both the matrices U and V are unitary (i.e. have orthonormal columns s.t. $V^*V = I$) and S is the diagonal matrix with (sorted decreasing) singular values σ_i. Multiplying A with an input vector along the first column in V gives

$$A \cdot V_{(:,1)} = USV^* \cdot V_{(:,1)} =$$
$$= US \begin{bmatrix} 1 \\ 0 \end{bmatrix} = U_{(:,1)} \cdot \sigma_1$$

That is, we get maximal gain σ_1 in the output direction $U_{(:,1)}$ if we use an input in direction $V_{(:,1)}$ (and minimal gain σ_2 if we use the second column $V_{(:,n)} = V_{(:,2)}$).

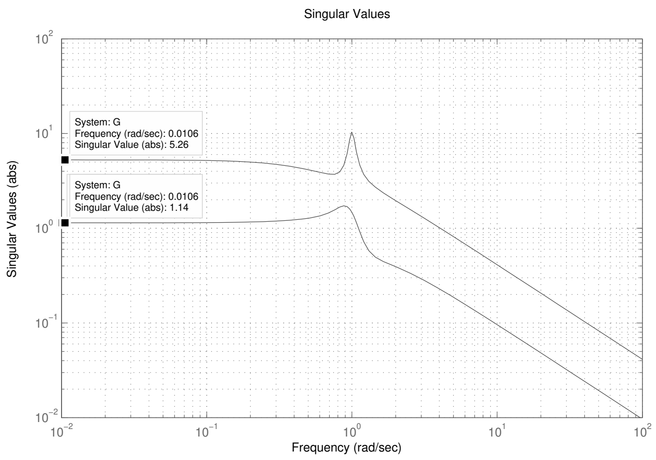
```
>> A = [2 4; 0 3]
A =
     2     4
     0     3
>> [U,S,V] = svd(A)
U =
    0.8416   -0.5401
    0.5401    0.8416
S =
    5.2631         0
         0    1.1400
V =
    0.3198   -0.9475
    0.9475    0.3198

>> A*V(:,1)
ans =
    4.4296
    2.8424

>> U(:,1)*S(1,1)
ans =
    4.4296
    2.8424
```


Example: Gain of multivariable system

Consider the transfer function matrix

$$G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{4}{2s+1} \\ \dfrac{s}{s^2+0.1s+1} & \dfrac{3}{s+1} \end{bmatrix}$$

```
>> s=zpk('s')
>> G=[ 2/(s+1) 4/(2*s+1); s/(s^2+0.1*s+1) 3/(s+1)];
>> sigma(G) % plot sigma values of G wrt freq
>> grid on
>> norm(G,inf)  % infinity norm = system gain
   ans =
      10.3577
```



The singular values of the tranfer function matrix (prev slide). Note that $G(0)$= [2 4 ; 0 3] which corresponds to A in the SVD-example above. $\|G\|_\infty = 10.3577$.

Lecture 2 – summary

- Input–output stability: $\|\mathcal{S}\| < \infty$
- Sensitivity function: $S := \frac{dT/T}{dP/P} = \frac{1}{1+PC}$
- Small Gain Theorem: The feedback interconnection of $\mathcal{S}_1$ and $\mathcal{S}_2$ is stable **if** $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$
  - Conservative compared to the Nyquist criterion
  - Useful for robustness analysis
- The gain of a multivariable system $G(s)$ is given by $\sup_\omega \overline{\sigma}(G(i\omega))$, where $\overline{\sigma}$ is the largest singular value