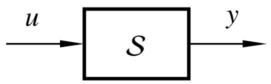
Lecture 1 – Outline

1. Course program
2. Course introduction
3. Signals and systems
   - System representations
   - Signal norm and system gain

Systems



A **system** is a mapping from the input signal $u(t)$ to the output signal $y(t)$, $-\infty < t < \infty$:

$$y = \mathcal{S}(u)$$

System properties

A system $\mathcal{S}$ is

- **causal** if $y(t_1)$ only depends on $u(t)$, $-\infty < t \leq t_1$, **non-causal** otherwise
- **static** if $y(t_1)$ only depends on $u(t_1)$, **dynamic** otherwise
- **discrete-time** if $u(t)$ and $y(t)$ are only defined for a countable set of discrete time instances $t = t_k,\ k = 0, \pm 1, \pm 2, \ldots$, **continuous-time** otherwise

System properties (cont'd)

A system $\mathcal{S}$ is

- **single-variable** or **scalar** if $u(t)$ and $y(t)$ are scalar signals, **multivariable** otherwise
- **time-invariant** if $y(t) = \mathcal{S}(u(t))$ implies $y(t+\tau) = \mathcal{S}(u(t+\tau))$, **time-varying** otherwise
- **linear** if $\mathcal{S}(\alpha_1 u_1 + \alpha_2 u_2) = \alpha_1 \mathcal{S}(u_1) + \alpha_2 \mathcal{S}(u_2)$, **nonlinear** otherwise

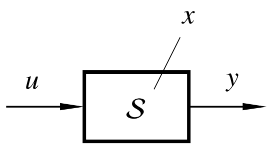
LTI system representations

We will mainly deal with continuous-time **linear time-invariant** (LTI) systems in this course

For LTI systems, the same input–output mapping $\mathcal{S}$ can be represented in a number of equivalent ways:

- linear ordinary differential equation
- linear state-space model
- transfer function
- impulse response
- step response
- frequency response
- ...

State-space models



Linear state-space model:

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Solution:

$$y(t) = Ce^{At}x(0) + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

Mini-problem 1

$$\begin{aligned} \dot{x}_1 &= -x_1 + 2x_2 + u_1 + u_2 - u_3 \\ \dot{x}_2 &= -5x_2 + 3u_2 + u_3 \\ y_1 &= x_1 + x_2 + u_3 \\ y_2 &= 4x_2 + 7u_1 \end{aligned}$$

How many states, inputs and outputs?

Determine the matrices A, B, C, D to write the system as

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Mini-problem 1

Change of coordinates

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

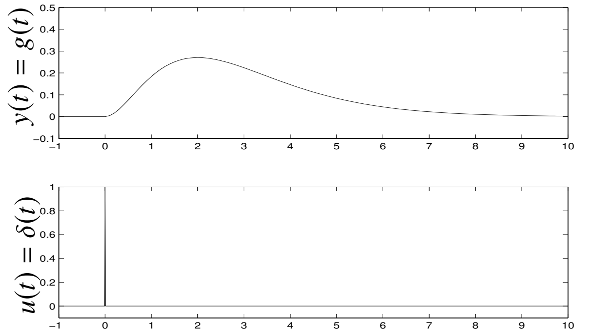
Change of coordinates

$$z = Tx, \quad T \text{ invertible}$$

$$\begin{cases} \dot{z} = T\dot{x} = T(Ax + Bu) & = T(AT^{-1}z + Bu) = TAT^{-1}z + TBu \\ y = Cx + Du & = CT^{-1}z + Du \end{cases}$$

Note: There are infinitely many different state-space representations of the same input–output mapping $y = \mathcal{S}(u)$

Impulse response

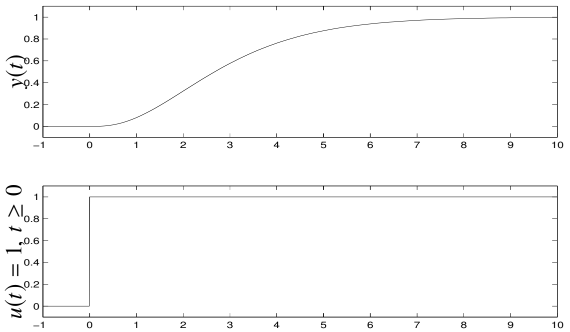


Common experiment in medicine and biology

$$g(t) = \int_0^t Ce^{A(t-\tau)}B\delta(\tau)d\tau + D\delta(t) = Ce^{At}B + D\delta(t)$$

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = (g * u)(t)$$

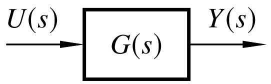
Step response



Common experiment in process industry

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)d\tau$$

Transfer function



$$G(s) = \mathcal{L}\{g(t)\}$$

$$y(t) = (g * u)(t) \quad \Leftrightarrow \quad Y(s) = G(s)U(s)$$

Conversion from state-space form to transfer function:

$$G(s) = C(sI - A)^{-1}B + D$$

Transfer function

A transfer function is **rational** if it can be written as

$$G(s) = \frac{B(s)}{A(s)}$$

where $B(s)$ and $A(s)$ are polynomials in s

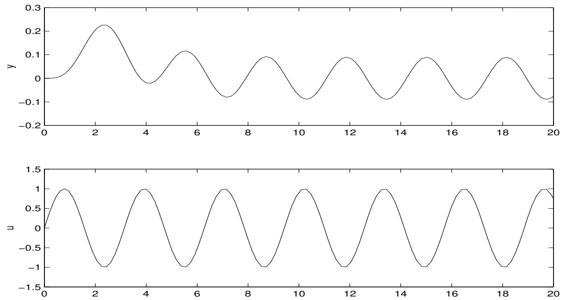
- Example of non-rational function: Time delay e^{-sL}

It is **proper** if $\deg B \leq \deg A$ and **strictly proper** if $\deg B < \deg A$

- Example of non-proper function: Pure derivative s

A rational and proper transfer function can be converted to state-space form (see Collection of Formulae)

Frequency response



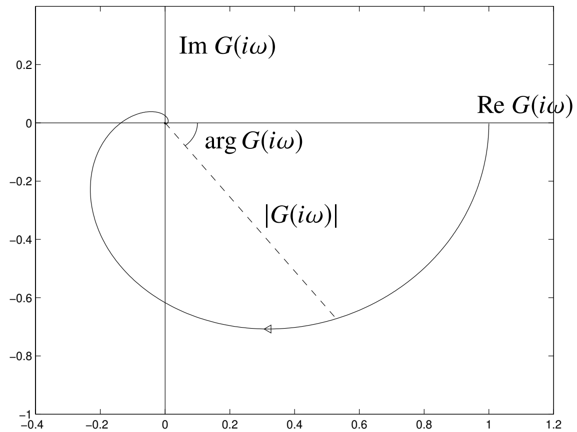
Assume stable transfer function $G = Lg$. Input $u(t) = \sin \omega t$ gives

$$y(t) = \int_0^t g(\tau)u(t-\tau)d\tau = \text{Im}\left[\int_0^t g(\tau)e^{-i\omega\tau}d\tau \cdot e^{i\omega t}\right]$$

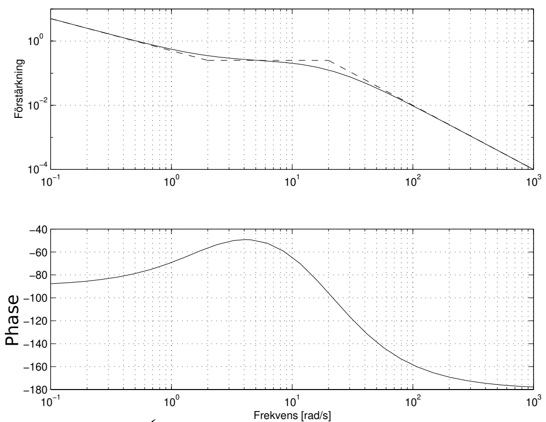
$$[t \to \infty] = \text{Im}\left(G(i\omega)e^{i\omega t}\right) = |G(i\omega)| \sin\left(\omega t + \arg G(i\omega)\right)$$

After a transient, also the output becomes sinusoidal

The Nyquist diagram



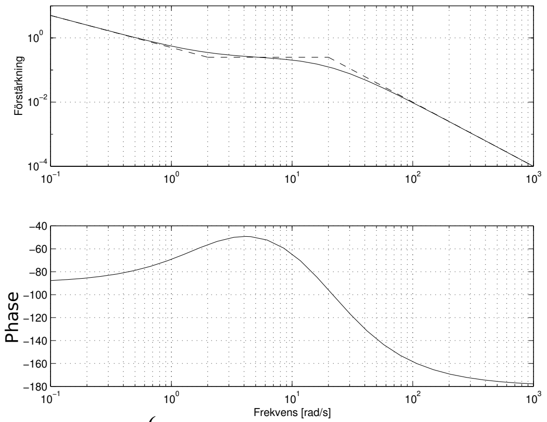
The Bode diagram



$$G = G_1 G_2 G_3 \qquad \begin{cases} \log|G| = \log|G_1| + \log|G_2| + \log|G_3| \\ \arg G = \arg G_1 + \arg G_2 + \arg G_3 \end{cases}$$

Each new factor enters additively!

The Bode diagram



$G = G_1 G_2 G_3$ $\begin{cases} \log|G| = \log|G_1| + \log|G_2| + \log|G_3| \\ \arg G = \arg G_1 + \arg G_2 + \arg G_3 \end{cases}$

Each new factor enters additively!

Hint: Set Matlab units

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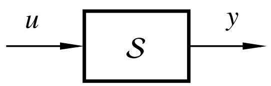
Signal norm and system gain



How to quantify

- the “size” of the signals u and y
- the “maximum amplification” between u and y

Signal norm

The L_2 norm of a signal $y(t) \in \mathbb{R}^n$ is defined as

$$\|y\| = \sqrt{\int_0^\infty |y(t)|^2 dt}$$

By Parseval’s theorem it can also be expressed as

$$\|y\| = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega}$$

System gain

The L_2 (or “L_2-induced”) gain of a general system $\mathcal{S}$ with input u and output $\mathcal{S}(u)$ is defined as

$$\|\mathcal{S}\| := \sup_u \frac{\|\mathcal{S}(u)\|}{\|u\|}$$

L_2 gain of LTI systems

THEOREM 1.1
Consider a stable LTI system $\mathcal{S}$ with transfer function $G(s)$. Then

$$\|\mathcal{S}\| = \sup_{\omega} |G(i\omega)| := \|G\|_\infty$$

Proof. Let $y = \mathcal{S}(u)$. Then

$$\|y\|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |Y(i\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |G(i\omega)|^2 |U(i\omega)|^2 d\omega \le \|G\|_\infty^2 \|u\|^2$$

The inequality is arbitrarily tight when $u(t)$ is a sinusoid near the maximizing frequency.

(How to interpret $|G(i\omega)|$ for matrix transfer functions will be explained in Lecture 2.)

Mini-problem 2

What are the L_2 gains of the following scalar LTI systems?

1. $y(t) = -u(t)$ (a sign shift)
2. $y(t) = u(t-T)$ (a time delay)
3. $y(t) = \int_0^t u(\tau) d\tau$ (an integrator)
4. $y(t) = \int_0^t e^{-(t-\tau)} u(\tau) d\tau$ (a first order filter)

Mini-problem 2

Lecture 1 – Summary

- Course overview
- Review of LTI system descriptions (see also Exercise 1)
- L_2 norm of signals
  - Definition: $\|y\| := \sqrt{\int_0^\infty |y(t)|^2 dt}$
- L_2 gain of systems
  - Definition: $\|\mathcal{S}\| := \sup_u \frac{\|\mathcal{S}(u)\|}{\|u\|}$
  - Special case—stable LTI systems: $\|\mathcal{S}\| = \sup_\omega |G(i\omega)| := \|G\|_\infty$ (also known as the "H_∞ norm" of the system)