Course Summary

L1–L5 Specifications, models and loop shaping by hand

L6–L8 Limitations on achievable performance

L9–L11 Controller optimization: Analytic approach

L12–L14 Controller optimization: Numerical approach

Some Real-World Examples

Flexible servo resonant system

Quadruple tank system multivariable (MIMO), RHP zero

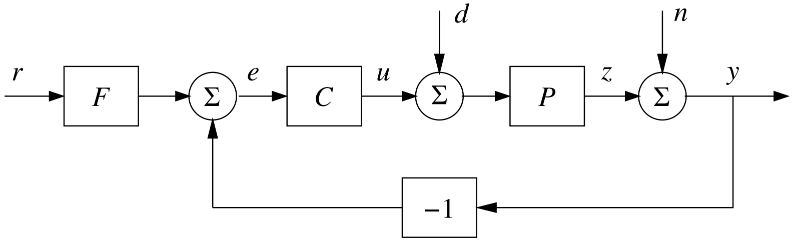
MinSeg robot multivariable (MISO), RHP pole

DVD focus control resonant system, wide frequency range, (midranging)

Bicycle steering unstable pole/zero-pair

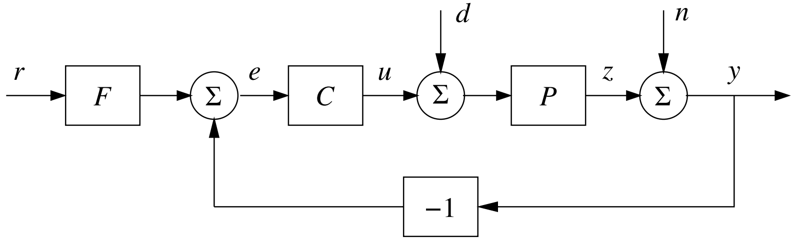
Ball in hoop zero in origin

2-DOF control



- Reduce the effects of load disturbances
- Limit the effects of measurement noise
- Reduce sensitivity to process variations
- Make output follow command signals

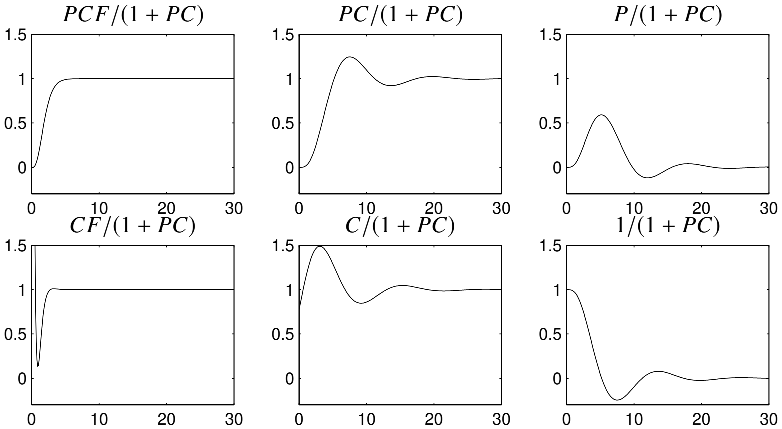
2DOF control



$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

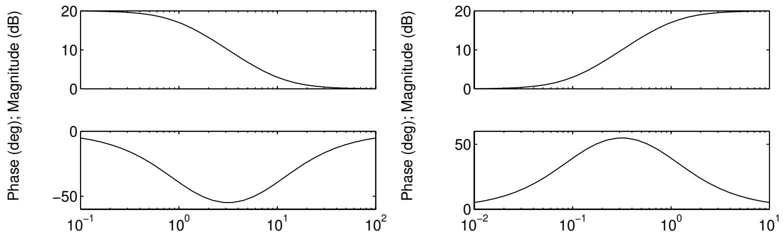
$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

Important step responses



Lag and lead filters for loop shaping

$$C(s) = \frac{s+10}{s+1} \qquad C(s) = \frac{10(s+1)}{(s+10)}$$



MIMO systems

If C, P and F are general MIMO-systems, so called **transfer function matrices**, the **order of multiplication matters** and

$$PC \neq CP$$

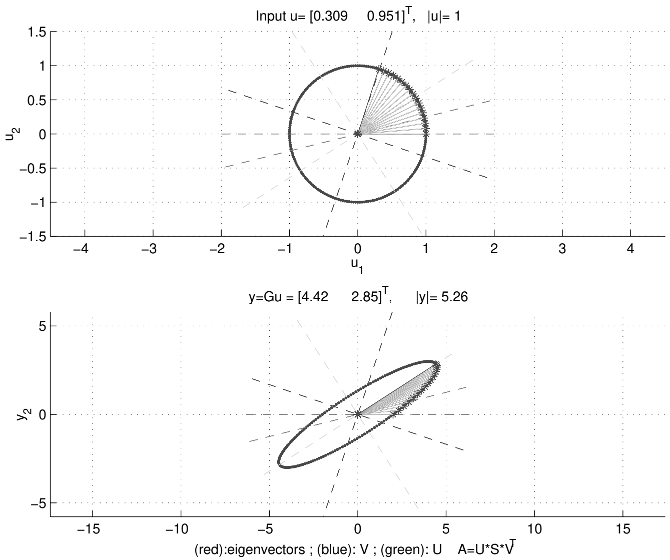
and thus we need to multiply with the inverse from the correct side as in general

$$(I+L)^{-1}M \neq M(I+L)^{-1}$$

Note, however that

$$(I+PC)^{-1}PC = P(I+CP)^{-1}C = PC(I+PC)^{-1}$$

Different gains in different directions: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



Realization of multivariable system

Example: To find state space realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+3)} \\ \frac{6}{(s+2)(s+4)} & \frac{1}{s+2} \end{bmatrix}$$

we write the transfer matrix as

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+3} \\ \frac{3}{s+2} - \frac{3}{s+4} & \frac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}}{s+1} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}3 & 1\end{bmatrix}}{s+2} - \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & 1\end{bmatrix}}{s+3} - \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}3 & 0\end{bmatrix}}{s+4}$$

This gives the realization

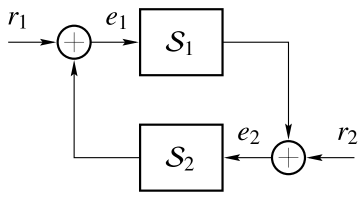
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$

The Small Gain Theorem

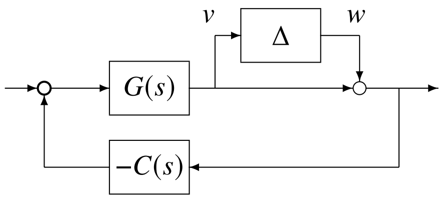
Consider a linear system $\mathcal{S}$ with input u and output $\mathcal{S}(u)$ having a (Hurwitz) stable transfer function $G(s)$. Then, the system gain

$$\|\mathcal{S}\| := \sup_u \frac{\|\mathcal{S}(u)\|}{\|u\|} \quad \text{is equal to} \quad \|G\|_\infty := \sup_\omega |G(i\omega)|$$



Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are input-output stable. If $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$, then the gain from (r_1, r_2) to (e_1, e_2) in the closed-loop system is finite.

Application to robustness analysis



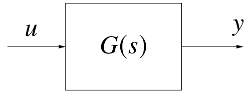
The transfer function from w to v is

$$-\frac{G(s)C(s)}{1 + G(s)C(s)}$$

Hence the small gain theorem guarantees closed-loop stability for all perturbations Δ with

$$\|\Delta\| < \left(\sup_\omega \left| \frac{G(i\omega)C(i\omega)}{1 + G(i\omega)C(i\omega)} \right| \right)^{-1}$$

Spectral density



Assume that the stationary mean-zero stochastic process u has spectral density $\Phi_u(\omega)$. Then

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G(i\omega)^*$$

- “Any spectrum” can be generated by filtering white noise
- Finding $G(s)$ given $\Phi_y(\omega)$ is called spectral factorization

State-space system with white noise input

Given the system

$$\dot{x} = Ax + Bw, \qquad \Phi_w(\omega) = R$$

the stationary covariance of the state x is given by

$$\mathrm{E}\, xx^T = \Pi_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega)d\omega$$

The symmetric matrix Π_x can be found by solving the Lyapunov equation

$$A\Pi_x + \Pi_x A^T + BRB^T = 0$$

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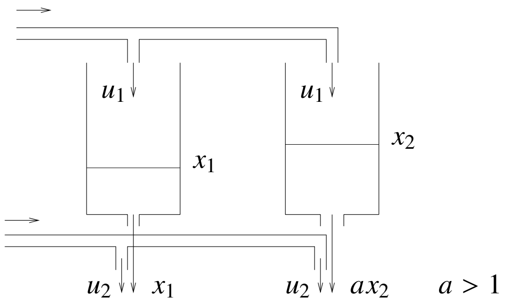
- ○ Specifications, models and loop shaping
- ● **Limitations on achievable performance**
- ○ Controller optimization: Analytic approach
- ○ Controller optimization: Numerical approach

Example: Two water tanks

Example from Lecture 6:

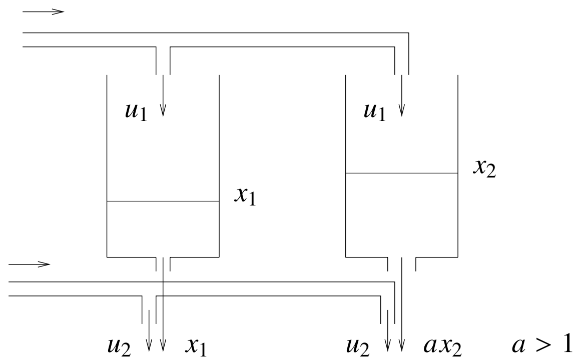


$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

$$y_1 = x_1 + u_2 \qquad y_2 = ax_2 + u_2$$

Can you reach $y_1 = 1, y_2 = 2$? Can you stay there?

Example: Two water tanks

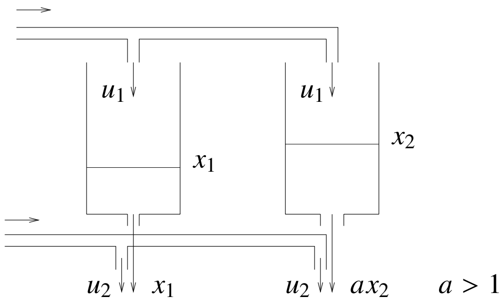


$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

The controllability Gramian $W_c = \int_0^{\infty} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$

is close to singular for $a \approx 1$, so it is harder to reach a desired state.

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix}. \quad \text{Find zero from} \quad \det G(s) = \frac{-s}{(s+1)(s+2)}$$

There is a zero at $s = 0$! Outputs must be equal at stationarity.

Sensitivity bounds from RHP zeros and poles

Rules of thumb:

“The cross-over frequency (or closed-loop bandwidth) must be smaller than unstable zero location z.”
“The cross-over frequency (or closed-loop bandwidth) must be greater than unstable pole location p.”

Hard bounds:

The sensitivity must be one at an unstable zero:

$$P(z) = 0 \qquad \Rightarrow \qquad S(z) := \frac{1}{1 + P(z)C(z)} = 1$$

The complimentary sensitivity must be one at an unstable pole:

$$P(p) = \infty \qquad \Rightarrow \qquad T(p) := \frac{P(p)C(p)}{1 + P(p)C(p)} = 1$$

Maximum Modulus Principle

Assume that $G(s)$ is rational, proper and stable. Then

$$\sup_{\omega \in \mathbb{R}} |G(i\omega)| \geq |G(s)|$$

for all s in the RHP.

Corollary:

Suppose that the plant $P(s)$ has unstable zeros z_i and unstable poles p_j. Then the specifications

$$\sup_{\omega} |W_S(i\omega)S(i\omega)| \leq 1 \qquad \sup_{\omega} |W_T(i\omega)T(i\omega)| \leq 1$$

are impossible to meet with a stabilizing controller unless $|W_S(z_i)| \leq 1$ for every unstable zero z_i and $|W_T(p_j)| \leq 1$ for every unstable pole p_j.

Relative Gain Array (RGA)

For a square matrix $A \in \mathbb{C}^{n \times n}$, define

$$\text{RGA}(A) := A \,.\!*\, (A^{-1})^T$$

where “$.*$” denotes element-by-element multiplication.
(For a non-square matrix, use pseudo inverse $A^\dagger$)

- The sum of all elements in a column or row is one.
- Permutations of rows or columns in A give the same permutations in RGA(A)
- RGA(A) is independent of scaling
- If A is triangular, then RGA(A) is the unit matrix I.

Example: RGA for a distillation column

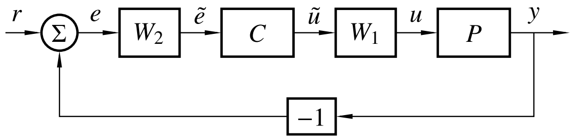
For pairing of inputs and outputs,

- select pairings that have relative gains close to 1.
- avoid pairings that have negative relative gain.

$$\text{RGA}(P(0)) = \begin{bmatrix} 0.2827 & -0.6111 & 1.3285 \\ 0.0134 & 1.5827 & -0.5962 \end{bmatrix}$$

To choose control signal for y_1, we apply the heuristics to the top row and choose u_3. Based on the bottom row, we choose u_2 to control y_2. Decentralized control!

Decoupling



Select decoupling filters W_1 (input decoupling) and/or W_2 (output decoupling) so that the controller sees a diagonal plant:

$$\tilde{P} = W_2 P W_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a decentralized controller C with the same diagonal structure.

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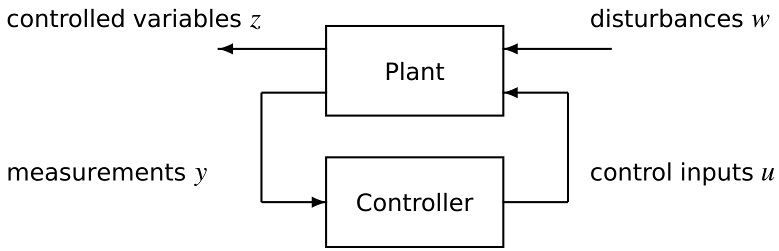
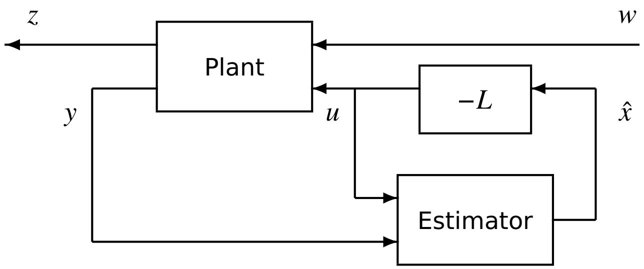
A general optimization setup



The objective is to find a controller that optimizes the transfer matrix $G_{zw}(s)$ from disturbances w to controlled outputs z.

Lectures 9–11: Problems with analytic solutions
Lectures 12–14: Problems with numeric solutions

Output feedback using state estimates



Plant: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + w_1(t) \\ y(t) = Cx(t) + w_2(t) \end{cases}$

Controller: $\begin{cases} \frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$

Linear Quadratic Gaussian (LQG) control

Given the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + w_1(k) \\ y(t) = Cx(t) + w_2(t) \end{cases} \qquad Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} > 0 \qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix} > 0$$

consider controllers of the form $u = -L\hat{x}$ with $\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K[y - C\hat{x}]$. The cost function

$$\mathrm{E}\left\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right\}$$

is minimized when K and L satisfy

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \qquad L = Q_2^{-1}(SB + Q_{12})^T$$

$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T \qquad K = (PC^T + R_{12})R_2^{-1}$$

Tuning the weights

- A small Q_2 compared to Q_1 means that control is “cheap”
  - Resulting LQ controller will have large feedback gain
  - The plant state will be quickly regulated to zero
  - A large cost on an individual state x_i means that more effort will be spent on regulating that particular state to zero
- A small R_2 compared to R_1 means that measurements can be trusted
  - Resulting Kalman filter will have large filter gain
  - The initial estimation error will quickly converge to zero
  - A large noise covariance on an individual state x_i means that the estimation error will decay faster for that particular state

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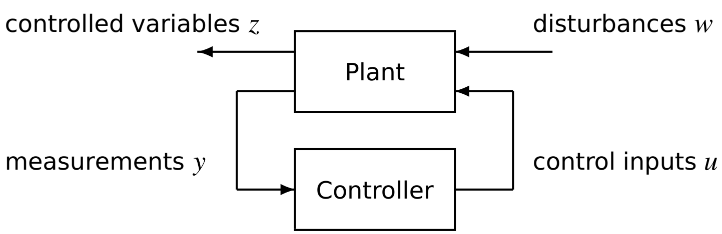
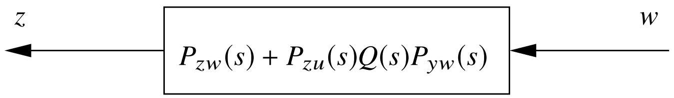
The Q-parameterization (Youla)

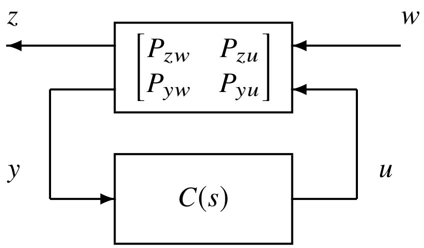


Idea for lecture 12-14:
The choice of controller generally corresponds to finding $Q(s)$, to get desirable properties of the map from w to z:



Once $Q(s)$ is determined, a corresponding controller is derived.

The Youla Parameterization



The closed-loop transfer matrix from w to z is

$$G_{zw}(s) = P_{zw}(s) + P_{zu}(s)Q(s)P_{yw}(s)$$

where

$$Q(s) = C(s)\left[I - P_{yu}(s)C(s)\right]^{-1}$$
$$C(s) = Q(s) - Q(s)P_{yu}(s)C(s)$$
$$C(s) = \left[I + Q(s)P_{yu}(s)\right]^{-1}Q(s)$$

Synthesis by convex optimization

A general control synthesis problem can be stated as a convex optimization problem in the variables $Q_0, \ldots, Q_m$. The problem has a quadratic objective, with linear and quadratic constraints:

Minimize $\int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega) \overbrace{\sum_k Q_k \phi_k(i\omega)}^{Q(i\omega)} P_{yw}(i\omega)|^2 d\omega$ } quadratic objective

subject to
step response $w_i \to z_j$ is smaller than f_{ijk} at time t_k
step response $w_i \to z_j$ is bigger than g_{ijk} at time t_k } linear constraints
Bode magnitude $w_i \to z_j$ is smaller than h_{ijk} at ω_k } quadratic constraints

Once the variables $Q_0, \ldots, Q_m$ have been optimized, the controller is obtained as $C(s) = \left[I + Q(s)P_{yu}(s)\right]^{-1}Q(s)$

Model reduction by balanced truncation

Consider a balanced realization

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + Du$$

with the lower part of the Gramian being $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_n)$.

Replacing the second state equation by $\dot{\hat{x}}_2 = 0$ gives the relation $0 = A_{21}\hat{x}_1 + A_{22}\hat{x}_2 + B_2 u$. The reduced system

$$\begin{cases} \dot{\hat{x}}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\hat{x}_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_r = (C_1 - C_2A_{22}^{-1}A_{21})\hat{x}_1 + (D - C_2A_{22}^{-1}B_2)u \end{cases}$$

satisfies the error bound

$$\|G - G_r\|_\infty = \frac{\|y - y_r\|_2}{\|u\|_2} \le 2(\sigma_{r+1} + \cdots + \sigma_n)$$