FRTN10 Multivariable Control, Lecture 14

Automatic Control LTH, 2017

Course Outline

Lecture 14 – Outline

1. Model reduction by balanced truncation
2. Application to controller simplification

[Glad/Ljung, section 3.6]

Model reduction

- Mathematical modeling can lead to dynamical models of very high order
- Controller synthesis using the Q-parameteritzation can lead to very high order controllers

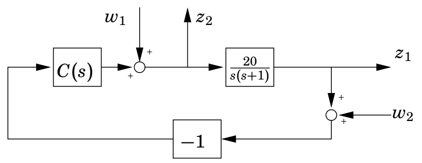
Need for systematic way to reduce the model order

In general terms we would like to achieve

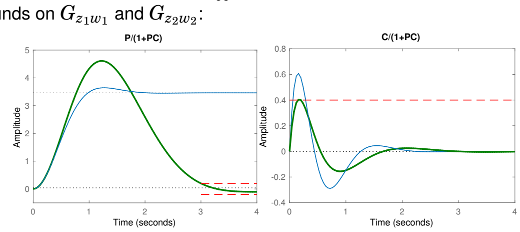
$$G_{\text{red}}(s) \approx G(s)$$

where $G_{\text{red}}(s)$ has (much) lower order than $G(s)$

Example – DC-motor



In Lecture 13 we minimized $\int_{-\infty}^{\infty} |G_{zw}(i\omega)|^2 d\omega$ subject to step response bounds on $G_{z_1w_1}$ and $G_{z_2w_2}$:



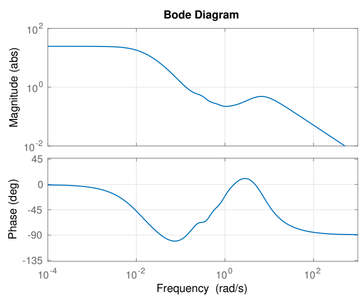
Example – DC-motor

Recall that

$$C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1} Q(s), \text{ with } Q(s) = \sum_{k=0}^{N} Q_k \phi_k(s).$$

Controller order grows with the number of basis functions ϕ_k.

Optimized controller for DC-servo has order 14. Is that really needed?



Controllability and Observability Gramians

System:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Impulse response from zero intial condition: $u_i(t) = \delta(t),\ x(0) = 0$

$$x(t) = e^{At}B$$
$$S = \int_0^\infty x(t)x^T(t)\,dt = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt$$

Output when $u = 0$ (only initial state $x(0) = x_0$)

$$y(t) = Cx(t) = Ce^{At}x_0$$
$$\int_0^\infty y(t)^Ty(t)dt = \int_0^\infty x_0^Te^{A^Tt}C^TCAtx_0dt \quad = \quad x_0^TOx_0$$

Controllability and Observability Gramians

The controllability Gramian $S = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ can be found by solving the Lyapunov equation

$$AS + SA^T + BB^T = 0$$

The observability Gramian $O = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$ can be found by solving the Lyapunov equation

$$A^TO + OA + C^TC = 0$$

We want to remove states that are both poorly controllable and poorly observable.

Idea: Introduce state transformation $\hat{x} = Tx$ that reveals these states

Balanced realizations

For a stable system (A, B, C) with Gramians S_x and O_x, the variable transformation $\hat{x} = Tx$ gives the new state-space matrices $\hat{A} = TAT^{-1}$, $\hat{B} = TB$, $\hat{C} = CT^{-1}$ and the new Gramians

$$S_{\hat{x}} = \int_0^\infty e^{\hat{A}t}\hat{B}\hat{B}^T e^{\hat{A}^T t} dt = \int_0^\infty T e^{At} B B^T e^{A^T t} T^T dt = T S_x T^T$$

$$O_{\hat{x}} = \int_0^\infty e^{\hat{A}^T t}\hat{C}^T \hat{C} e^{\hat{A}t} dt = \int_0^\infty T^{-T} e^{At} C^T C e^{A^T t} T^{-1} dt = T^{-T} O_x T^{-1}$$

A particular choice of T gives $S_{\hat{x}} = O_{\hat{x}} = \Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$

The corresponding realization $(\hat{A}, \hat{B}, \hat{C})$ is called a **balanced realization**.

Computing the balancing state transformation

(Not done by hand)

Compute the Cholesky decompositions

$$S_x = WW^T, \quad O_x = ZZ^T$$

and the singular value decomposition

$$W^T Z = U \Sigma V^T$$

The balancing transformation is then given by

$$T = \Sigma^{-\frac{1}{2}} V^T Z^T, \quad T^{-1} = W U \Sigma^{-\frac{1}{2}}$$

Matlab: `[sysb,sigmas,T] = balreal(sys)`

Hankel singular values

Notice that

$$\begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} = \underbrace{(T S_x T^T)}_{\Sigma} \underbrace{(T^{-T} O_x T^{-1})}_{\Sigma} = T S_x O_x T^{-1}$$

so the diagonal elements are the eigenvalues of $S_x O_x$, independently of coordinate system.

The numbers $\sigma_1, \ldots, \sigma_n$ are called the **Hankel singular values** of the system.

A small Hankel singular value corresponds to a state that is both weakly controllable and weakly observable. Hence, it can be truncated without much effect on the input-output behavior.

Model reduction by balanced truncation

Consider a balanced realization

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + Du$$

with the lower part of the Gramian being $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_n)$.

Replacing the second state equation by $\dot{\hat{x}}_2 = 0$ gives the relation $0 = A_{21}\hat{x}_1 + A_{22}\hat{x}_2 + B_2 u$. The reduced system

$$\begin{cases} \dot{\hat{x}}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\hat{x}_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_{\text{red}} = (C_1 - C_2 A_{22}^{-1} A_{21})\hat{x}_1 + (D - C_2 A_{22}^{-1} B_2)u \end{cases}$$

satisfies the error bound

$$\frac{\|y - y_{\text{red}}\|_2}{\|u\|_2} \le 2\sigma_{r+1} + \cdots + 2\sigma_n$$

Example

Original system: $\quad G(s) = \dfrac{1 - s}{s^6 + 3s^5 + 5s^4 + 7s^3 + 5s^2 + 3s + 1}$

Hankel singular values:

$$\{\sigma_i\} = \begin{bmatrix} 1.9837 & 1.9184 & 0.7512 & 0.3292 & 0.1478 & 0.0045 \end{bmatrix}$$

Keeping $r = 3$ states gives the reduced system

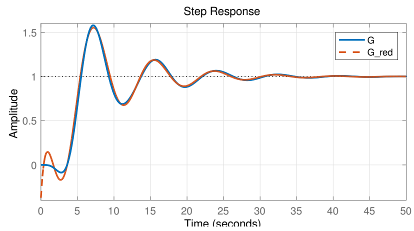
$$G_{\text{red}}(s) = \frac{0.3717s^3 - 0.9682s^2 + 1.14s - 0.5185}{s^3 + 1.136s^2 + 0.825s + 0.5185}$$

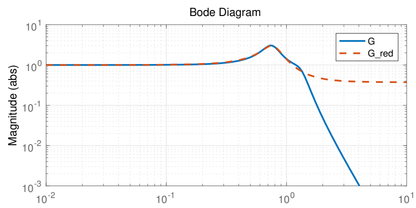
The error bound is

$$\frac{\|y - y_{\text{red}}\|_2}{\|u\|_2} \le 0.963$$

Matlab: `Gred = balred(G,3)`

Example





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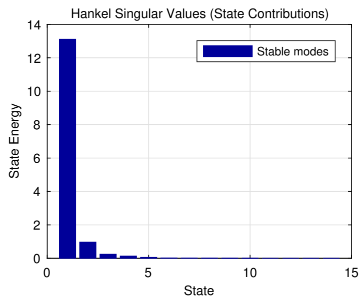
Example – DC-servo

Computing the 14 Hankel singular values gives

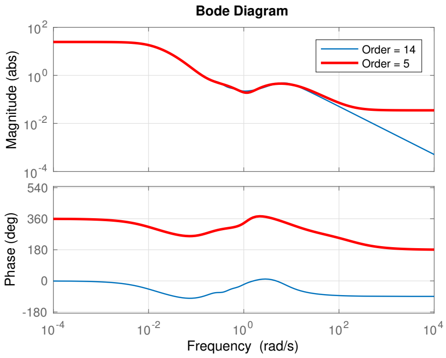
$$\begin{bmatrix} 13.11 & 0.97 & 0.24 & 0.14 & 0.05 & 0.02 & 0.01 & \ldots \end{bmatrix}$$



Matlab: `hsvd`

Example – DC-servo

Reduced controller with 5 states:



Matlab: `balred`

Handling unstable systems

Before model reduction, decompose the system into its stable and nonstable parts:

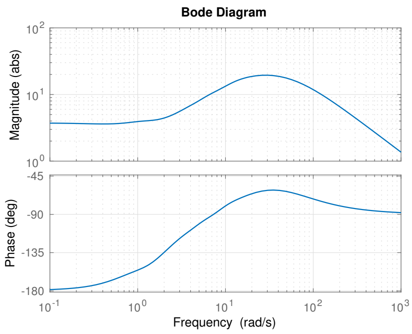
$$G(s) = G_s(s) + G_{ns}(s)$$

Perform the reduction only on $G_s(s)$; then add $G_{ns}(s)$ again

(Performed automatically by Matlab's `balreal` and `balred`)

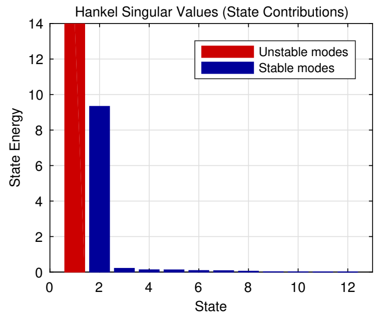
Example – Doyle–Stein (1979)

In Lecture 13 we found the following 12th order controller for Doyle–Stein's example using optimization:



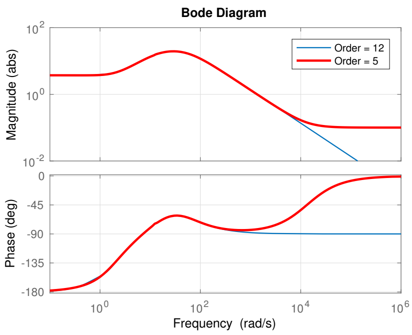
Example – Doyle–Stein (1979)

The controller has one unstable pole in 16.1. Hankel singular values:



Example – Doyle–Stein (1979)

Reduced controller with 5 states:



Summary

- Low-order controllers could be desirable to meet constraints on speed and memory
- Balanced realizations can reveal less important states
- Model reduction by balanced trunction has good theoretical error bounds
- Many possible extensions, e.g.
  - frequency weighting
  - reduction of unstable systems