FRTN10 Multivariable Control, Lecture 14

Automatic Control LTH, 2017

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

12. Youla parameterization, internal model control
13. Synthesis by convex optimization
14. **Controller simplification**

Lecture 14 – Outline

1. Model reduction by balanced truncation
2. Application to controller simplification

[Glad/Ljung, section 3.6]

Model reduction

- Mathematical modeling can lead to dynamical models of very high order
- Controller synthesis using the Q-parameteritzation can lead to very high order controllers

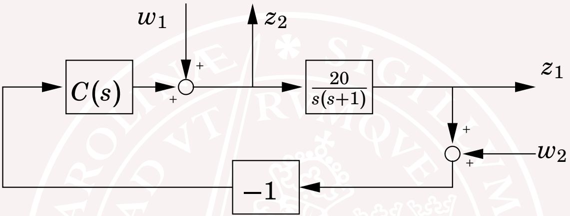
Need for systematic way to reduce the model order

In general terms we would like to achieve

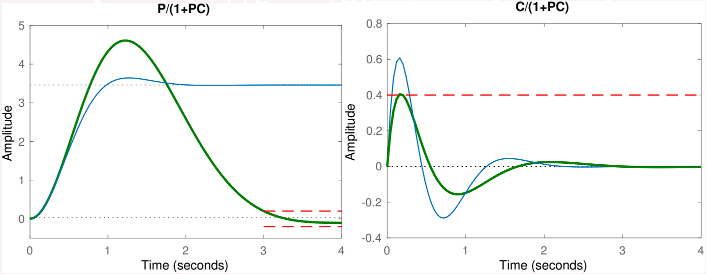
$$G_{\text{red}}(s) \approx G(s)$$

where $G_{\text{red}}(s)$ has (much) lower order than $G(s)$

Example – DC-motor



In Lecture 13 we minimized $\int_{-\infty}^{\infty} |G_{zw}(i\omega)|^2 d\omega$ subject to step response bounds on $G_{z_1 w_1}$ and $G_{z_2 w_2}$:



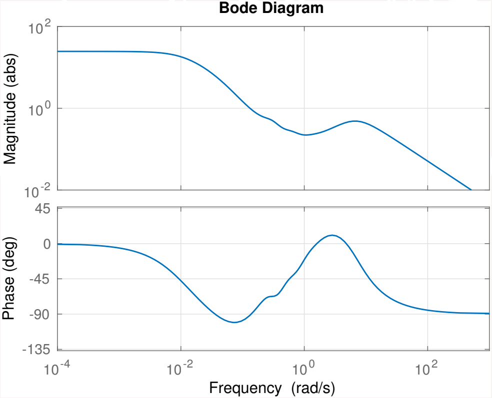
Example – DC-motor

Recall that

$$C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1} Q(s), \text{ with } Q(s) = \sum_{k=0}^{N} Q_k \phi_k(s).$$

Controller order grows with the number of basis functions ϕ_k.

Optimized controller for DC-servo has order 14. Is that really needed?



Controllability and Observability Gramians

System:

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

Impulse response from zero intial condition: $u_i(t) = \delta(t),\ x(0) = 0$

$$x(t) = e^{At}B$$
$$S = \int_0^\infty x(t)x^T(t)\,dt = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt$$

Output when $u = 0$ (only initial state $x(0) = x_0$)

$$y(t) = Cx(t) = Ce^{At}x_0$$

$$\int_0^\infty y(t)^Ty(t)dt = \int_0^\infty x_0^Te^{A^Tt}C^TCAtx_0dt \quad = \quad x_0^TOx_0$$

Controllability and Observability Gramians

The controllability Gramian $S = \int_0^\infty e^{At} BB^T e^{A^T t} dt$ can be found by solving the Lyapunov equation

$$AS + SA^T + BB^T = 0$$

The observability Gramian $O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ can be found by solving the Lyapunov equation

$$A^T O + OA + C^T C = 0$$

We want to remove states that are both poorly controllable and poorly observable.

Idea: Introduce state transformation $\widehat{x} = Tx$ that reveals these states

Balanced realizations

For a stable system (A, B, C) with Gramians S_x and O_x, the variable transformation $\widehat{x} = Tx$ gives the new state-space matrices $\widehat{A} = TAT^{-1}$, $\widehat{B} = TB$, $\widehat{C} = CT^{-1}$ and the new Gramians

$$S_{\widehat{x}} = \int_0^\infty e^{\widehat{A}t}\widehat{B}\widehat{B}^T e^{\widehat{A}^T t}dt = \int_0^\infty Te^{At}BB^T e^{A^T t}T^T dt = TS_xT^T$$

$$O_{\widehat{x}} = \int_0^\infty e^{\widehat{A}^T t}\widehat{C}^T\widehat{C}e^{\widehat{A}t}dt = \int_0^\infty T^{-T}e^{At}C^TCe^{A^T t}T^{-1}dt = T^{-T}O_xT^{-1}$$

A particular choice of T gives $S_{\widehat{x}} = O_{\widehat{x}} = \Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$

The corresponding realization $(\widehat{A}, \widehat{B}, \widehat{C})$ is called a **balanced realization**.

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Computing the balancing state transformation

(Not done by hand)

Compute the Cholesky decompositions

$$S_x = WW^T, \quad O_x = ZZ^T$$

and the singular value decomposition

$$W^T Z = U\Sigma V^T$$

The balancing transformation is then given by

$$T = \Sigma^{-\frac{1}{2}} V^T Z^T, \quad T^{-1} = WU\Sigma^{-\frac{1}{2}}$$

Matlab: `[sysb,sigmas,T] = balreal(sys)`

Hankel singular values

Notice that

$$\begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} = \underbrace{(TS_xT^T)}_{\Sigma}\underbrace{(T^{-T}O_xT^{-1})}_{\Sigma} = TS_xO_xT^{-1}$$

so the diagonal elements are the eigenvalues of S_xO_x, independently of coordinate system.

The numbers $\sigma_1, \ldots, \sigma_n$ are called the **Hankel singular values** of the system.

A small Hankel singular value corresponds to a state that is both weakly controllable and weakly observable. Hence, it can be truncated without much effect on the input-output behavior.

Model reduction by balanced truncation

Consider a balanced realization

$$\begin{bmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} + Du$$

with the lower part of the Gramian being $\Sigma_2 = \mathrm{diag}(\sigma_{r+1}, \ldots, \sigma_n)$.

Replacing the second state equation by $\dot{\hat{x}}_2 = 0$ gives the relation $0 = A_{21}\hat{x}_1 + A_{22}\hat{x}_2 + B_2 u$. The reduced system

$$\begin{cases} \dot{\hat{x}}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\hat{x}_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_{\mathrm{red}} = (C_1 - C_2 A_{22}^{-1}A_{21})\hat{x}_1 + (D - C_2 A_{22}^{-1}B_2)u \end{cases}$$

satisfies the error bound

$$\frac{\|y - y_{\mathrm{red}}\|_2}{\|u\|_2} \leq 2\sigma_{r+1} + \cdots + 2\sigma_n$$

Example

Original system: $$G(s) = \frac{1-s}{s^6 + 3s^5 + 5s^4 + 7s^3 + 5s^2 + 3s + 1}$$

Hankel singular values:

$$\{\sigma_i\} = \begin{bmatrix} 1.9837 & 1.9184 & 0.7512 & 0.3292 & 0.1478 & 0.0045 \end{bmatrix}$$

Keeping $r = 3$ states gives the reduced system

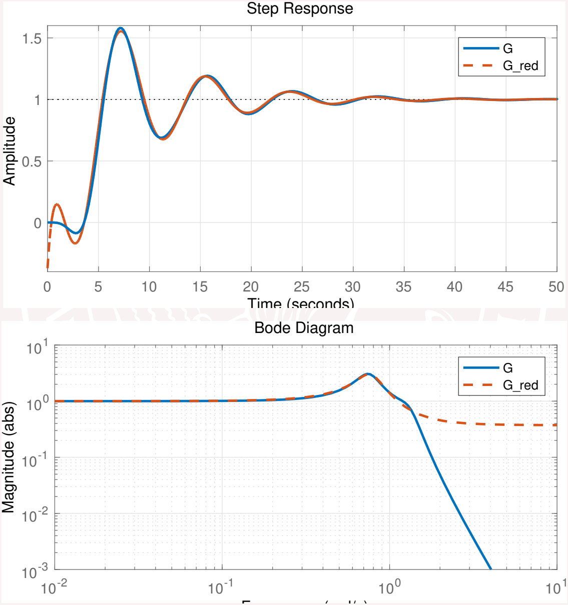
$$G_{\text{red}}(s) = \frac{0.3717s^3 - 0.9682s^2 + 1.14s - 0.5185}{s^3 + 1.136s^2 + 0.825s + 0.5185}$$

The error bound is

$$\frac{\|y - y_{\text{red}}\|_2}{\|u\|_2} \leq 0.963$$

Matlab: `Gred = balred(G,3)`

Example



Step Response
Amplitude
1.5
1
0.5
0
0
5
10
15
20
25
30
35
40
45
50
Time (seconds)
G
G_red
Bode Diagram
Magnitude (abs)
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
10^0
10^1
G
G_red

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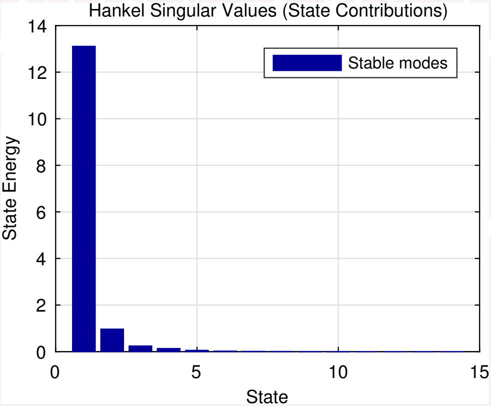
Example – DC-servo

Computing the 14 Hankel singular values gives

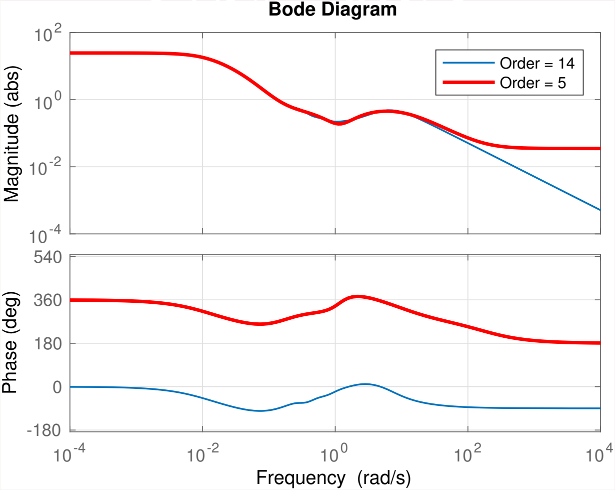
$$[13.11 \quad 0.97 \quad 0.24 \quad 0.14 \quad 0.05 \quad 0.02 \quad 0.01 \quad \ldots]$$



Matlab: `hsvd`

Example – DC-servo

Reduced controller with 5 states:



Matlab: `balred`

Handling unstable systems

Before model reduction, decompose the system into its stable and nonstable parts:

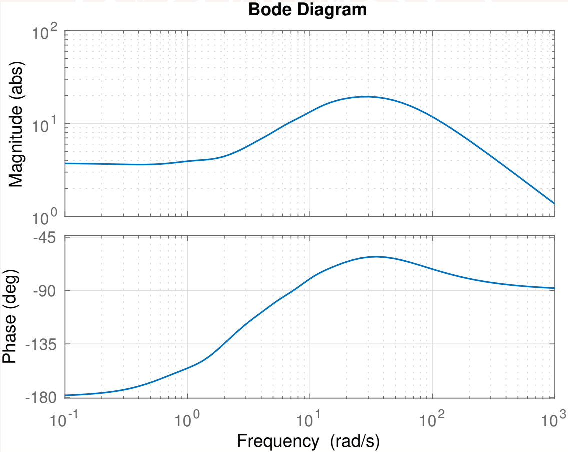
$$G(s) = G_s(s) + G_{ns}(s)$$

Perform the reduction only on $G_s(s)$; then add $G_{ns}(s)$ again

(Performed automatically by Matlab's `balreal` and `balred`)

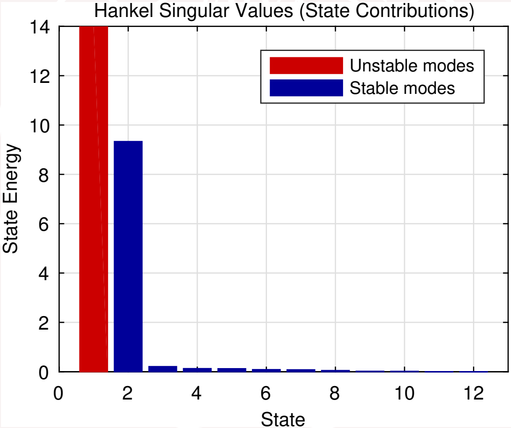
Example – Doyle–Stein (1979)

In Lecture 13 we found the following 12th order controller for Doyle–Stein's example using optimization:



Example – Doyle–Stein (1979)

The controller has one unstable pole in 16.1. Hankel singular values:



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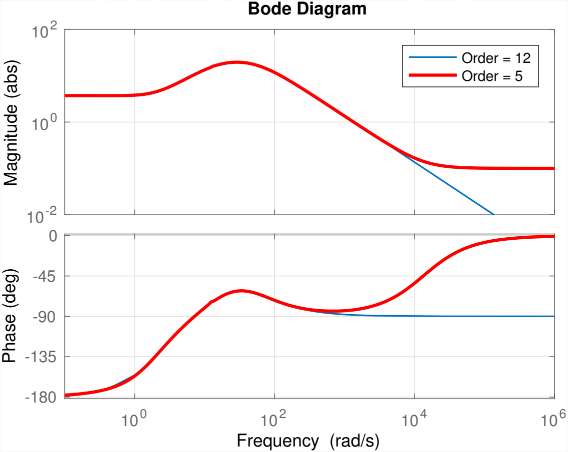

Summary

- Low-order controllers could be desirable to meet constraints on speed and memory
- Balanced realizations can reveal less important states
- Model reduction by balanced trunction has good theoretical error bounds
- Many possible extensions, e.g.
  - frequency weighting
  - reduction of unstable systems