

FRTN10 Multivariable Control, Lecture 10



Automatic Control LTH, 2017

Course Outline

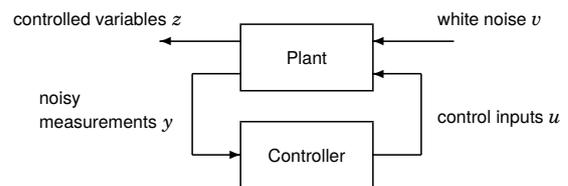
- L1-L5 Specifications, models and loop-shaping by hand
- L6-L8 Limitations on achievable performance
- L9-L11 Controller optimization: Analytic approach
 - 9. Linear-quadratic control
 - 10. **Kalman filtering, LQG**
 - 11. More on LQG
- L12-L14 Controller optimization: Numerical approach

Lecture 10 – Outline

1. Observer-based feedback
2. The Kalman filter
3. LQG

[Glad&Ljung sections 9.1–9.4 and 5.7]

Goal: Linear-quadratic-Gaussian control (LQG)



For a linear plant, let v be white noise of intensity R . Find a controller that minimizes the output variance

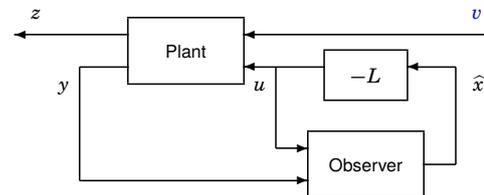
$$E|z|^2 = E \left\{ x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u \right\}$$

Previous lecture: State feedback solution ($y = x$, no meas. noise)

Lecture 10 – Outline

1. Observer-based feedback
2. The Kalman filter
3. LQG

Output feedback using an observer



$$\text{Plant: } \begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t) + Nv_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$$

$$\text{Controller: } \begin{cases} \frac{d\hat{x}(t)}{dt} = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$$

Closed-loop dynamics

Eliminate u and y :

$$\frac{dx(t)}{dt} = Ax(t) - BL\hat{x}(t) + Nv_1(t)$$

$$\frac{d\hat{x}(t)}{dt} = A\hat{x}(t) - BL\hat{x}(t) + K[Cx(t) - C\hat{x}(t)] + Kv_2(t)$$

Introduce the observer error $\tilde{x} = x - \hat{x}$

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix} + \begin{bmatrix} Nv_1(t) \\ Nv_1(t) - Kv_2(t) \end{bmatrix}$$

How to optimize the observer?

Lecture 10 – Outline

1. Observer-based feedback
2. **The Kalman filter**
3. LQG

Rudolf E. Kálmán, 1930–2016

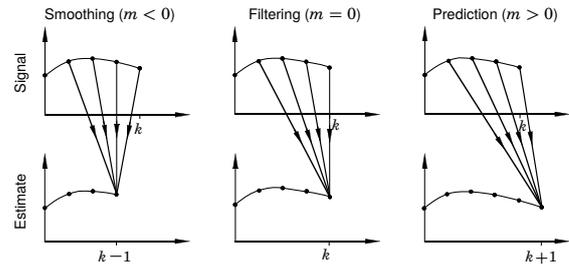


Recipient of the 2008 Charles Stark Draper Prize from the US National Academy of Engineering "for the development and dissemination of the optimal digital technique (known as the Kalman Filter) that is pervasively used to control a vast array of consumer, health, commercial and defense products."

Optimal filtering and prediction

- ▶ Wiener (1949): Stationary input-output formulation
- ▶ Kalman (1960): Time-varying state-space formulation (discrete time) ["A new approach to linear filtering and prediction problems", *Transactions of ASME—Journal of Basic Engineering*, **82**]

General problem: Estimate $x(k+m)$ given $\{y(i), u(i) \mid i \leq k\}$



Examples

- Smoothing** To estimate the Wednesday temperature based on measurements from Tuesday, Wednesday and Thursday
- Filtering** To estimate the Wednesday temperature based on measurements from Monday, Tuesday and Wednesday
- Prediction** To predict the Wednesday temperature based on measurements from Sunday, Monday and Tuesday

The optimal observer problem

The observer error dynamics are given by

$$\frac{d\tilde{x}}{dt} = (A - KC)\tilde{x} + \begin{pmatrix} N & -K \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

The noise $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ is assumed white with (co)intensity

$$\begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} > 0$$

Optimization problem: Assuming that the system is observable¹, find the gain K that minimizes the stationary error covariance

$$P = E \tilde{x} \tilde{x}^T$$

¹detectable is sufficient, see G&L

Finding the optimal observer gain

The stationary error covariance P is given by the Lyapunov equation

$$(A - KC)P + P(A - KC)^T + \begin{pmatrix} N & -K \end{pmatrix} \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} \begin{pmatrix} N^T \\ -K^T \end{pmatrix} = 0$$

Completing the square,

$$AP + PA^T + NR_1N^T + (KR_2 - PC^T - NR_{12})R_2^{-1}(KR_2 - PC^T - NR_{12})^T - (PC^T + NR_{12})R_2(PC^T + NR_{12}) = 0$$

we find that the minimum variance is attained for

$$K = (PC^T + NR_{12})R_2^{-1}$$

What remains is an algebraic Riccati equation,

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

The Kalman filter

[G&L Theorem 5.4]

Given an observable linear plant disturbed by white noise,

$$\begin{cases} \dot{x} = Ax + Bu + Nv_1 \\ y = Cx + v_2 \end{cases} \quad \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} > 0$$

the optimal observer is given by

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

where K is given by

$$K = (PC^T + NR_{12})R_2^{-1}$$

where $P = E(x - \hat{x})(x - \hat{x})^T > 0$ is the solution to

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

Remarks

The optimal observer gain does not depend on what state(s) we are interested in. The Kalman filter produces the optimal estimate of **all states** at the same time.

The optimal observer gain K is static since we are solving a steady-state problem.

(The Kalman filter can also be derived for finite-horizon problems and problems with time-varying system matrices. We then obtain a Riccati differential equation for $P(t)$ and a time-varying filter gain $K(t)$)

Duality between state feedback and state estimation

State feedback	State estimation
A	A^T
B	C^T
Q_1	NR_1N^T
Q_2	R_2
Q_{12}	NR_{12}
S	P
L	K^T

Kalman filter in Matlab (1)

lqe Kalman estimator design for continuous-time systems.

Given the system

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw && \{\text{State equation}\} \\ y &= Cx + Du + v && \{\text{Measurements}\} \end{aligned}$$

with unbiased process noise w and measurement noise v with covariances

$$E\{ww'\} = Q, \quad E\{vv'\} = R, \quad E\{wv'\} = N,$$

$[L,P,E] = \text{lqe}(A,G,C,Q,R,N)$ returns the observer gain matrix L such that the stationary Kalman filter

$$\dot{\hat{x}}_e = A\hat{x}_e + Bu + L(y - C\hat{x}_e - Du)$$

produces an optimal state estimate \hat{x}_e of x using the sensor measurements y . The resulting Kalman estimator can be formed with ESTIM.

Kalman filter in Matlab (2)

kalman Kalman state estimator.

$[\text{KEST},L,P] = \text{kalman}(\text{SYS},\text{QN},\text{RN},\text{NN})$ designs a Kalman estimator KEST for the continuous- or discrete-time plant SYS. For continuous-time plants

$$\begin{aligned} \dot{x} &= Ax + Bu + Gw && \{\text{State equation}\} \\ y &= Cx + Du + Hw + v && \{\text{Measurements}\} \end{aligned}$$

with known inputs u , process disturbances w , and measurement noise v , KEST uses $[u(t);y(t)]$ to generate optimal estimates $y_e(t),x_e(t)$ of $y(t),x(t)$ by:

$$\dot{\hat{x}}_e = A\hat{x}_e + Bu + L(y - C\hat{x}_e - Du)$$

$$\begin{aligned} |y_e| &= |C| \hat{x}_e + |D| u \\ |x_e| &= |I| \hat{x}_e + |0| u \end{aligned}$$

kalman takes the state-space model $\text{SYS}=\text{SS}(A,[B\ G],C,[D\ H])$ and the covariance matrices:

$$\text{QN} = E\{ww'\}, \quad \text{RN} = E\{vv'\}, \quad \text{NN} = E\{wv'\}.$$

Example 1 – Kalman filter for an integrator

$$\begin{aligned} \dot{x}(t) &= v_1(t) && v_1 \text{ is white noise with intensity } R_1 \\ y(t) &= x(t) + v_2(t) && v_2 \text{ is white noise with intensity } R_2 \end{aligned}$$

$$\frac{d\hat{x}}{dt} = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)]$$

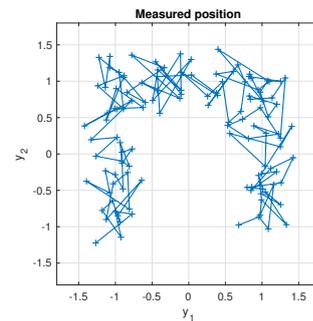
$$\text{Riccati equation} \quad 0 = R_1 - P^2/R_2 \Rightarrow P = \sqrt{R_1 R_2}$$

$$\text{Filter gain} \quad K = P/R_2 = \sqrt{R_1/R_2}$$

Interpretation?

Example 2 – Tracking of a moving object

Position readings $y = (y_1, y_2)^T$ with measurement noise:



Would like to estimate the true position

Example 2 – Tracking of a moving object

Dynamic model: Two double integrators driven by noise, $\ddot{y}_i = v_{1i}$

$$\text{State vector: } x = (\text{pos}_1 \quad \text{vel}_1 \quad \text{pos}_2 \quad \text{vel}_2)^T$$

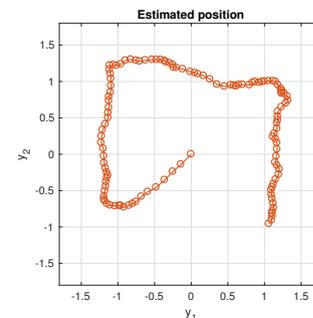
State-space model:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} v_1 \\ y &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} x + v_2 \end{aligned}$$

Fix $R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and design Kalman filter for different R_2

Example 2 – Tracking of a moving object

Simulation of Kalman filter from initial condition $\hat{x} = (0 \ 0)^T$

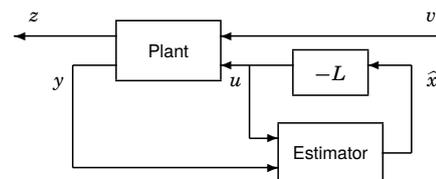


Larger R_2 gives better noise rejection but slower tracking

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Optimal output feedback – LQG



$$\text{Plant: } \begin{cases} \dot{x} = Ax + Bu + Nv_1 \\ y = Cx + v_2 \end{cases} \quad \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} > 0$$

$$\text{Controller: } \begin{cases} \frac{d}{dt} \hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$$

$$\text{Minimize}_{L,K} \quad E|z|^2 = E \left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u \right)$$

The separation principle

The following separation principle holds for linear systems with quadratic cost and Gaussian white noise disturbances:

- ▶ The optimal state feedback gain L is independent of the state uncertainty
- ▶ The optimal Kalman filter gain K is independent of the control objective

This makes it possible to optimize L and K separately.

[See G&L Theorem 9.1 and Corollary 9.1 for more details]

Example – LQG control of an integrator

Consider the problem to minimize $E(Q_1x^2 + Q_2u^2)$ for

$$\begin{cases} \dot{x}(t) = u(t) + v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases} \quad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

The observer-based controller

$$\begin{cases} \frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$$

is optimal with K and L computed as follows:

$$0 = Q_1 - S^2/Q_2 \Rightarrow S = \sqrt{Q_1Q_2} \Rightarrow L = S/Q_2 = \sqrt{Q_1/Q_2}$$

$$0 = R_1 - P^2/R_2 \Rightarrow P = \sqrt{R_1R_2} \Rightarrow K = P/R_2 = \sqrt{R_1/R_2}$$

Example – Control of a LEGO segway

Essentially an inverted pendulum – classical control problem



Sensors: Accelerometer, gyroscope

Actuators: DC motors

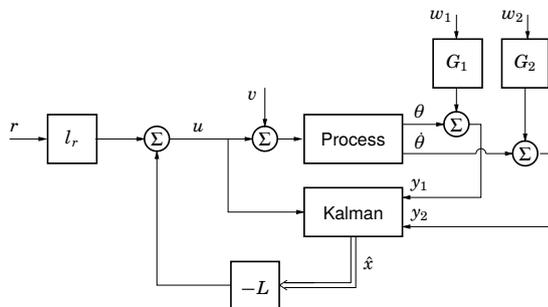
Sensor fusion

The two sensors have very different characteristics:

- ▶ The accelerometer is good for measuring the steady-state angle but very sensitive to disturbances at higher frequencies
- ▶ The gyroscope can measure the angular speed and track fast movements, but due to drift it cannot track the steady-state angle

Solution: Sensor fusion using Kalman filter

Modeling for LQG design



- ▶ v, w_1, w_2 – white noise sources
- ▶ $G_1(s) = \frac{s+a}{s/N+a}$ – models the inaccuracy of the accelerometer
- ▶ $G_2(s) = \frac{s+b}{s}$ – models the inaccuracy of the gyroscope

Lecture 10 – summary

- ▶ Observer-based feedback
- ▶ The Kalman filter – an optimal observer
- ▶ LQG by separation (LQ state feedback + Kalman filter)

Next lecture: More on LQG:

- ▶ Robustness of LQG?
- ▶ How to choose the design weights Q and R ?
- ▶ How to handle reference signals and integral action?
- ▶ Examples