FRTN10 Multivariable Control, Lecture 8

Automatic Control LTH, 2017

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

6. Controllability, observability, multivariable zeros
7. Fundamental limitations
8. **Multivariable and decentralized control**

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Lecture 8 – Outline

See “Lecture notes” and [G&L, Chapters 1, 7.7 (first part) and 8.3]

Typical process control system

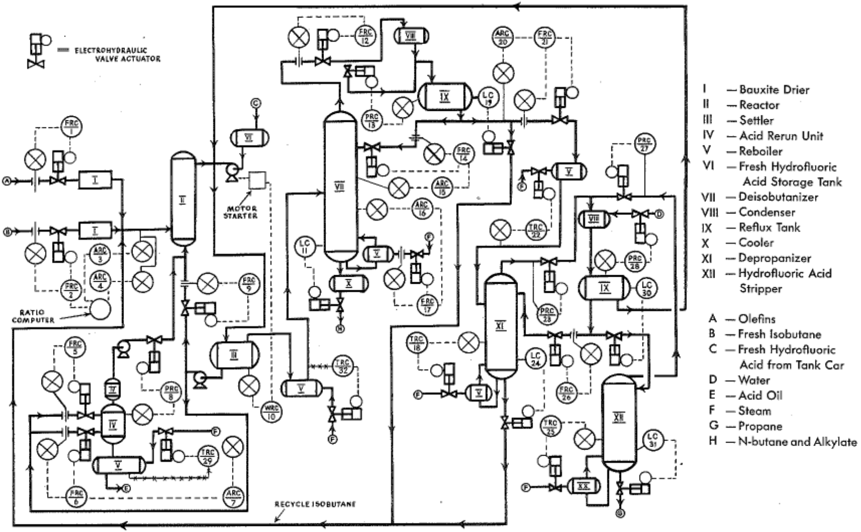
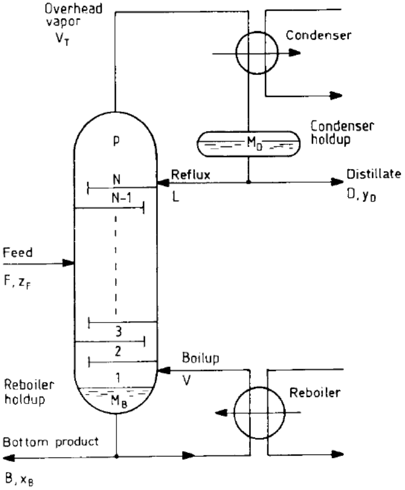


Figure 13–6. Automatic control system for Perco motor fuel alkylation process.

Example system: Distillation column



Raw oil inserted at bottom; different petro-chemical subcomponents extracted

Example system: Distillation column

Outputs:

$y_1 =$ top draw composition

$y_2 =$ side draw composition

Inputs:

$u_1 =$ top draw flowrate

$u_2 =$ side draw flowrate

$u_3 =$ bottom temperature control input

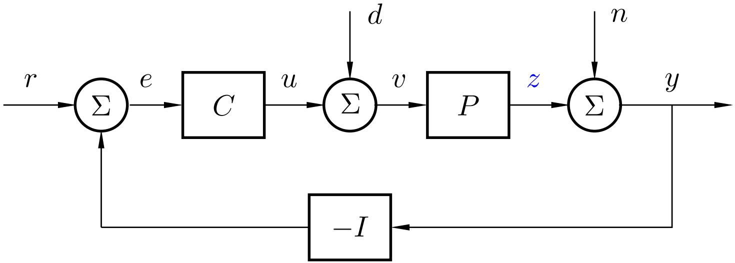
Linear first-order plus deadtime (FOPDT) model:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{4}{50s+1}e^{-27s} & \frac{1.8}{60s+1}e^{-28s} & \frac{5.9}{50s+1}e^{-27s} \\ \frac{5.4}{50s+1}e^{-18s} & \frac{5.7}{60s+1}e^{-14s} & \frac{6.9}{40s+1}e^{-15s} \end{bmatrix}}_{P(s)} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$$

Lecture 7 – Outline

1. Transfer functions for MIMO systems
2. Limitations due to RHP zeros
3. Decentralized control
4. Decoupling

Multivariable transfer functions



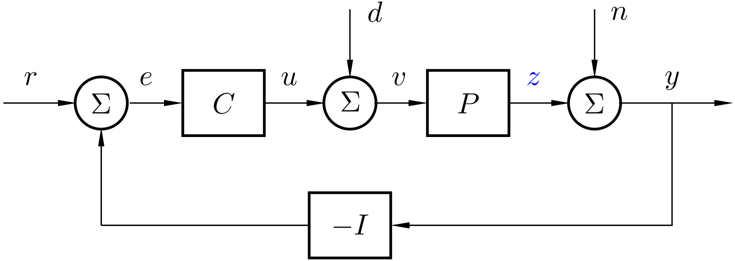
P and C are matrices and all signals are vectors – order matters!

$$Z = PCR + PD - PC(N + Z)$$

$$(I + PC)Z = PCR + PD - PCN$$

$$Z = \underbrace{(I + PC)^{-1}PC}_{G_{zr}=T}\ R + \underbrace{(I + PC)^{-1}P}_{G_{zd}}\ D - \underbrace{(I + PC)^{-1}PC}_{G_{zn}}\ N$$

Multivariable transfer functions



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Sensitivity functions for MIMO systems

Output sensitivity function:

$$(I + PC)^{-1} = S$$

Input sensitivity function:

$$(I + CP)^{-1}$$

Mini-problem:

Find the transfer functions above in the block diagram on the previous slide.

Some useful math relations

Notice the following identities:

$$(i)\quad [I+PC]^{-1}P = P[I+CP]^{-1}$$

$$(ii)\quad C[I+PC]^{-1} = [I+CP]^{-1}C$$

$$(iii)\quad T = P[I+CP]^{-1}C = PC[I+PC]^{-1} = [I+PC]^{-1}PC$$

$$(iv)\quad S+T = I$$

Proof:

The first equality follows by multiplication on both sides with $(I+PC)$ from the left and with $(I+CP)$ from the right.

Left: $[I+PC][I+PC]^{-1}P[I+CP] = I\cdot[P+PCP] = [I+PC]P$

Right: $[I+PC]P[I+CP]^{-1}[I+CP] = [I+PC]P\cdot I = [I+PC]P$

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Hard limitations from RHP zeros

[G&L Theorem 7.9]

Assume that the MIMO system $P(s)$ has a transmission zero z in the RHP.

Let $S(s) = [I + P(s)C(s)]^{-1}$ and let $W_S(s)$ be a scalar, stable and minimum phase transfer function. Then the specification

$$\|W_S S\|_\infty = \sup_\omega \bar{\sigma}\big(W_S(i\omega)S(i\omega)\big) \leq 1$$

is only possible to meet if

$$|W_S(z)| \leq 1$$

Example: Control of MIMO system with RHP zero

[G&L Example 1.1]

Process:

$$P(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Computing the determinant

$$\det P(s) = \frac{2}{(s+1)^2} - \frac{3}{(s+2)(s+1)} = \frac{-s+1}{(s+1)^2(s+2)}$$

shows that the process has a RHP zero in 1, which will limit the achievable performance.

[See lecture notes for details of the following slides]

Example – Controller 1

The controller

$$C_1(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & -\frac{3K_2(s+0.5)}{s(s+2)} \\ -\frac{K_1(s+1)}{s} & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

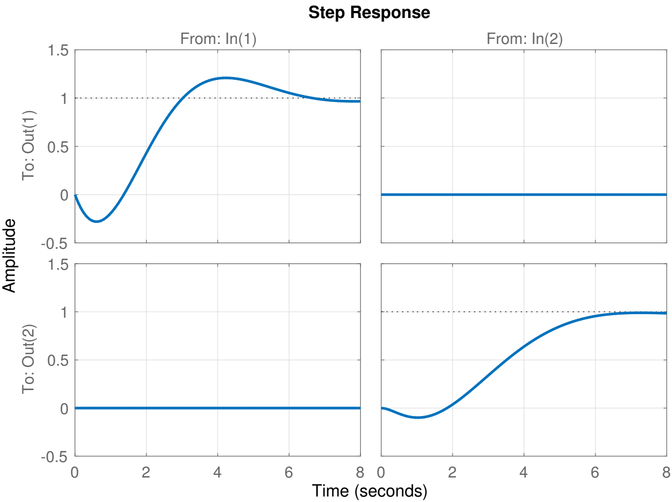
gives the diagonal loop transfer matrix

$$P(s)C_1(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & 0 \\ 0 & \frac{K_2(s+0.5)(-s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

The system is decoupled into two scalar loops, each with an unstable zero at $s = 1$ that limits the bandwidth.

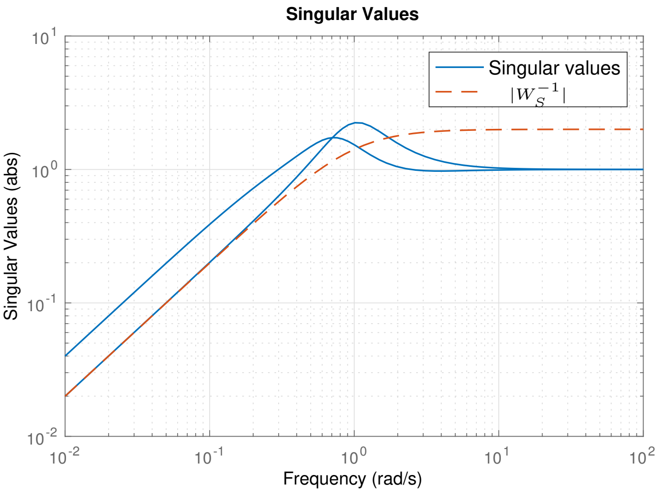
Closed-loop step responses from (r_1, r_2) to (y_1, y_2) for $K_1 = K_2 = 1$ are shown on next slide.

Step responses using Controller 1



No cross-coupling, but RHP zero shows up in both y_1 and y_2.

Sensitivity sigma plot using Controller 1



$W_S(s) = \frac{s+1}{2s}$, impossible to meet due to RHP zero

Example – Controller 2

The controller

$$C_2(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & K_2 \\ -\frac{K_1(s+1)}{s} & K_2 \end{bmatrix}$$

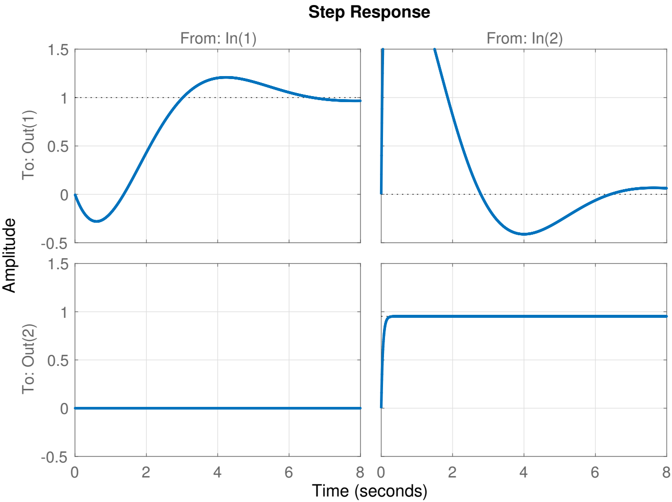
gives the triangular loop transfer matrix

$$P(s)C_2(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & \frac{K_2(5s+7)}{(s+2)(s+1)} \\ 0 & \frac{2K_2}{s+1} \end{bmatrix}$$

Now the decoupling is only partial: Output y_2 is not affected by r_1. Moreover, there is no RHP zero that limits the rate of response in y_2!

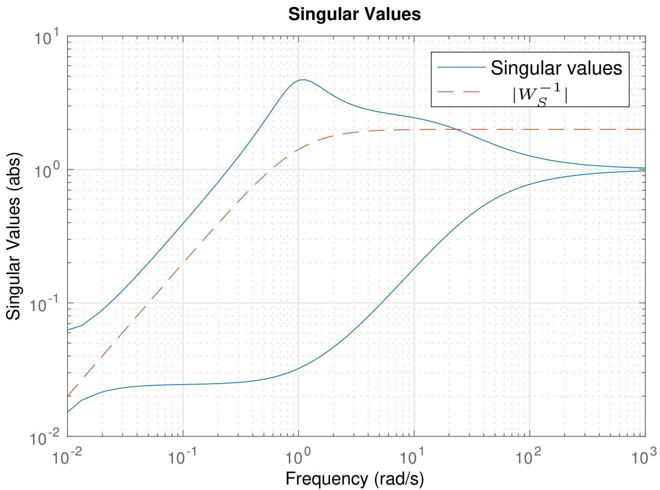
The closed loop step responses for $K_1 = 1$, $K_2 = 10$ are shown on next slide.

Step responses using Controller 2



The RHP zero does not prevent a fast y_2 response to r_2 but at the price of a simultaneous undesired response in y_1.

Sensitivity sigma plot using Controller 2



$W_S(s) = \frac{s+1}{2s}$, impossible to meet due to RHP zero

Example – Controller 3

The controller

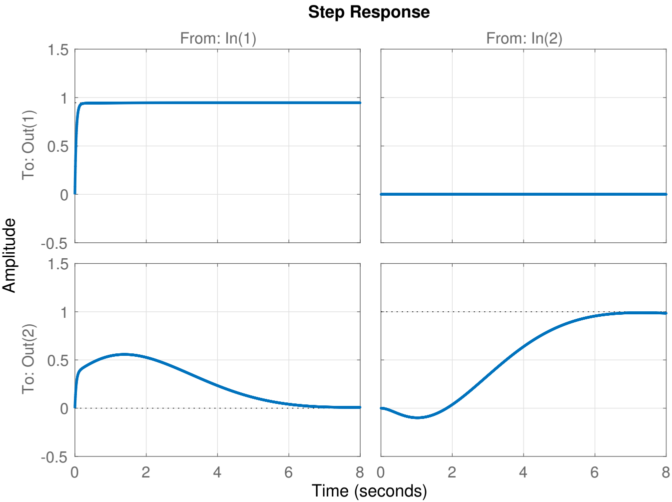
$$C_3(s) = \begin{bmatrix} K_1 & \frac{-3K_2(s+0.5)}{s(s+2)} \\ K_1 & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

gives the triangular loop transfer matrix

$$P(s)C_3(s) = \begin{bmatrix} \frac{K_1(5s+7)}{(s+1)(s+2)} & 0 \\ \frac{2K_1}{s+1} & \frac{K_2(-1+s)(s+0.5)}{s(s+1)^2(s+2)} \end{bmatrix}$$

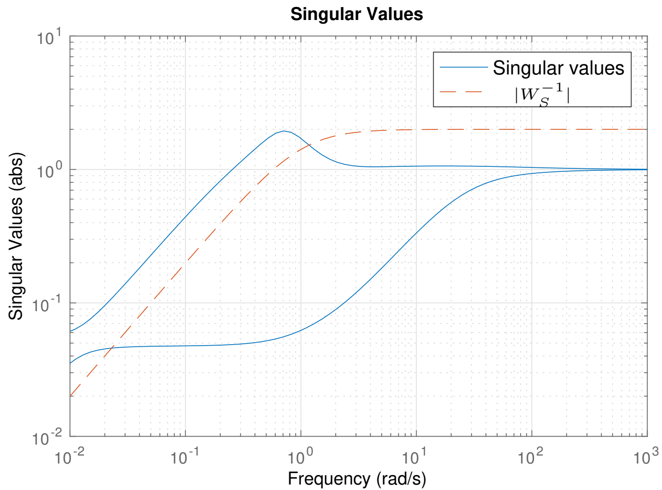
In this case y_1 is decoupled from r_2 and can respond arbitrarily fast for high values of K_1, at the expense of bad behavior in y_2. Step responses for $K_1 = 10$, $K_2 = 1$ are shown on next slide.

Step responses using Controller 3



The RHP zero does not prevent a fast y_1 response to r_1 but at the price of a simultaneous undesired response in y_2.

Sensitivity sigma plot using Controller 3



$W_S(s) = \frac{s+1}{2s}$, impossible to meet due to RHP zero

Example – summary

To summarize, the example shows that even though a **multivariable RHP zero always gives a performance limitation**, it is **possible to influence** where the effects should show up.

Lecture 7 – Outline

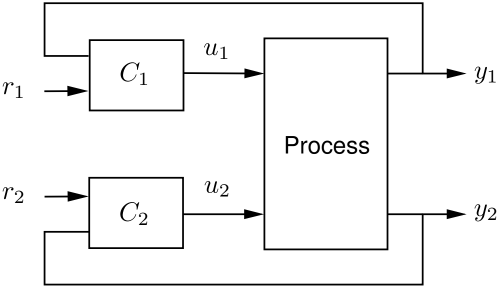
Decentralized control

Background in process control:

- A few important variables were controlled using the simple loop paradigm: one sensor, one actuator, one controller
- As more loops were added, interaction was handled using feedforward, cascade and midrange control, selectors, etc.
- Not always obvious how to associate sensors and actuators – the pairing problem

Computer control and state-space design methods eventually led to centralized MIMO control schemes (LQG, MPC, etc.)

Interaction between simple loops



$$Y_1(s) = P_{11}(s)U_1(s) + P_{12}U_2(s)$$
$$Y_2(s) = P_{21}(s)U_1(s) + P_{22}U_2(s),$$

What happens when the controllers are tuned individually (C_1 for P_{11} and C_2 for P_{22}), ignoring the cross-couplings?

Rosenbrock's example

$$P(s) = \begin{pmatrix} \dfrac{1}{s+1} & \dfrac{2}{s+3} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{pmatrix}$$

Very benign subsystems (compare with example in [G&L, Ch.1]).

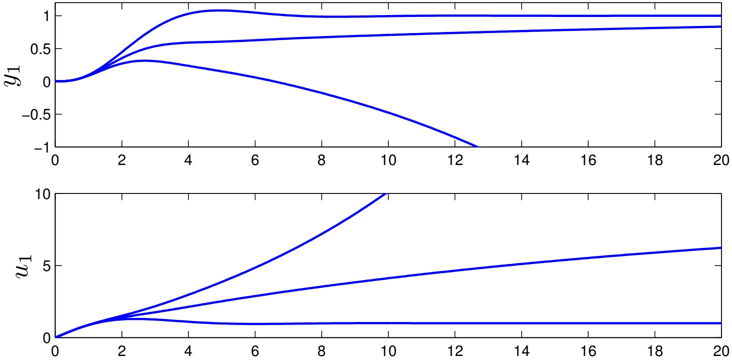
The transmission zeros are given by the roots of

$$\det P(s) = \frac{1}{s+1}\Big(\frac{1}{s+1} - \frac{2}{s+3}\Big) = \frac{1-s}{(s+1)^2(s+3)}$$

RHP zero in 1 $\Rightarrow$ cannot robustly control the system with a crossover frequency larger than 1.

Rosenbrock's example with two SISO controllers

- $U_1 = \left(1 + \frac{1}{s}\right)(R_1 - Y_1)$
- $U_2 = -K_2 Y_2$ with $K_2 = 0$, 0.8, and 1.6.



The second controller has a major impact on the first loop! Gain reversal in $u_1 \to y_1$ when $K_2 = 1.6$.

Bristol's relative gain array (RGA)

- Edgar H. Bristol, "On a new measure of interaction for multivariable process control" [*IEEE TAC* 11(1967) pp. 133–135]
- A simple way of measuring interaction in MIMO systems
- Idea: Study how the gain between one input and one output changes when all other outputs are regulated:

$$\text{relative gain} = \frac{\text{open-loop gain}}{\text{closed-loop gain}}$$

- Often only the static gain $P(0)$ is analyzed, but one could also look at for instance $P(i\omega_c)$

Calculation of RGA

Assume a square MIMO system with input-output relation $y = Gu$.

Open loop: Assume $u_j \neq 0$ and all other inputs zero. This gives

$$y = G_{*j} u_j$$

Output k is given by

$$y_k = G_{kj} u_j$$

Closed loop: Assume $y_k \neq 0$ and that all other outputs are regulated to zero. Solving for the corresponding inputs gives

$$u = G^{-1}_{*k} y_k$$

Input j is given by

$$u_j = G^{-1}_{jk} y_k \quad \Leftrightarrow \quad y_k = \frac{1}{G^{-1}_{jk}} u_j$$

Calculation of RGA

Ratio of open-loop and closed-loop gain:

$$\lambda_{kj} = G_{kj} \cdot G_{jk}^{-1}$$

All elements of the relative gain array (matrix) can be computed as

$$\Lambda = \mathrm{RGA}(G) = G \mathbin{.*} (G^{-1})^T$$

where $.*$ denotes element-wise (Hadamard/Schur) multiplication

Matlab: `RGA = G.*inv(G).'`

Properties and interpretation of RGA

- RGA is dimensionless; not affected by choice of units or scaling.
- RGA is normalized: Rows and columns of Λ sum to 1.
- Diagonal or triangular plant gives $\Lambda = I$

Interpretation:

- $\lambda_{kj} \approx 1$ means small closed-loop interaction. Suitable to pair output k with input j.
- $\lambda_{kj} < 0$ corresponds to a sign reversal due to feedback and a risk of instability if output k is paired with input j – avoid!
- $0 < \lambda_{kj} < 1$ means that the closed-loop gain is larger than the open-loop gain; the opposite is true for $\lambda_{kj} > 1$.

Recommendation: Pair the outputs and inputs so that corresponding relative gains are positive and as close to 1 as possible.

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RGA of Rosenbrock's example

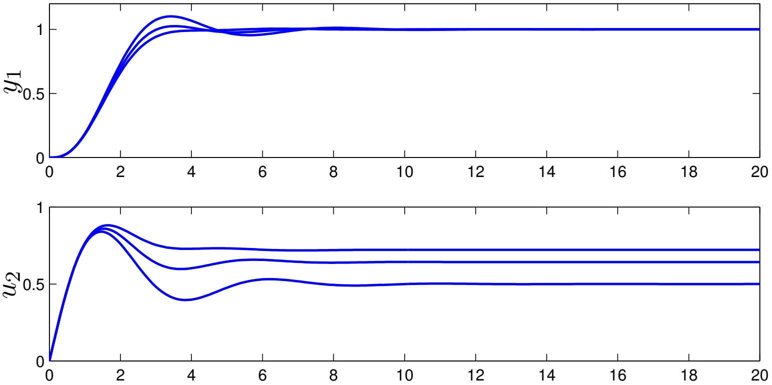
Analysis of static gain:

$$P(0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad P^{-1}(0) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\Lambda = P(0) .* (P^{-1}(0))^T = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix}$$

- Negative value of λ_{11} indicates the problematic sign reversal found previously when y_1 was controlled using u_1.
- Better to use reverse pairing, i.e. let u_2 control y_1 and vice versa.

Rosenbrock's example with reverse pairing



- $U_2 = \left(1 + \frac{1}{s}\right)(R_1 - Y_1)$
- $U_1 = -K_2 Y_2$ with $K_2 = 0, 0.8$, and 1.6.

RGA of non-square systems

The RGA can also be computed for a general gain matrix G:

$$\mathrm{RGA}(G) = G \,.\!*\, \left(G^{\dagger}\right)^{T}$$

Here, $\dagger$ denotes the pseudo-inverse (Matlab: `pinv`)

Example: Distillation column:

$$P(0) = \begin{pmatrix} 4.0 & 1.8 & 5.9 \\ 5.4 & 5.7 & 6.9 \end{pmatrix}, \quad \mathrm{RGA}(P(0)) = \begin{pmatrix} 0.28 & -0.61 & 1.33 \\ 0.01 & 1.58 & -0.59 \end{pmatrix}$$

Suggested pairing for decentralized control: y_1–u_3, y_2–u_2, u_1 unused

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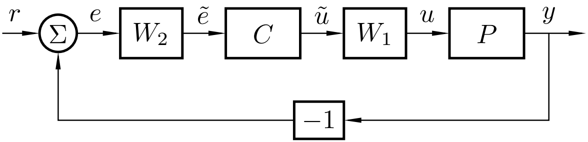


Idea: Select decoupling filters W_1 and W_2 so that the controller sees a diagonal plant:

$$\tilde{P} = W_2 P W_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a decentralized controller C with the same diagonal structure.

Decoupling

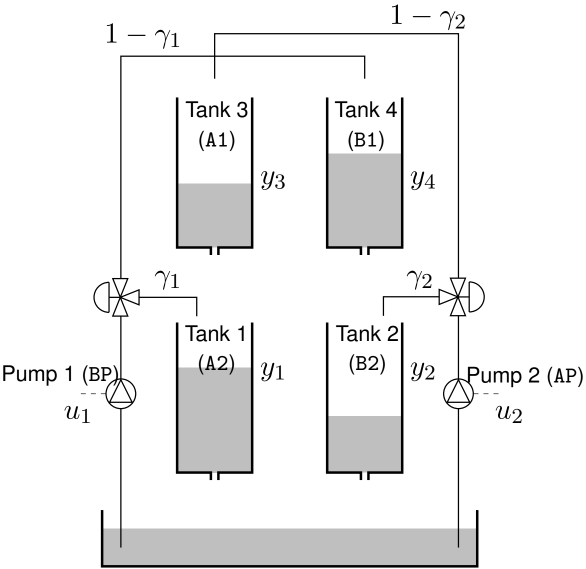
Many variants/names:

- Input/conventional/feedforward decoupling: $\tilde{P} = PW_1,\ W_2 = I$
- Output/inverse/feedback decoupling: $\tilde{P} = W_2 P,\ W_1 = I$

W_1 and W_2 can be static or dynamic systems

Example: Static input decoupling: $W_1 = P^{-1}(0),\ W_2 = I$

Lab 2: The quadruple tank



Summary

- All real systems are coupled
- Multivariable RHP zeros $\Rightarrow$ limitations
  - Don’t forget process redesign
- Decentralized control – one controller per controlled variable
  - RGA gives insight for input–output pairing
- Decoupling

  Simpler system

  SISO design, tuning and operation can be used

Next week: Centralized multivariable design using LQ/LQG