FRTN10 Multivariable Control, Lecture 7

Automatic Control LTH, 2017

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

6. Controllability, observability, multivariable zeros
7. **Fundamental limitations**
8. Multivariable and decentralized control

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Lecture 7 – Outline

1. Bode's Relation and Bode's Integral Theorem
2. Limitations from unstable poles, RHP zeros and delays: Intuition
3. Limitations from unstable poles and RHP zeros: Hard proofs

[Glad & Ljung: 7.2–7.9]

Limitations in control design

What we already know:

- Model errors, measurement noise, control signal limitations $\Rightarrow$ upper limit on achievable bandwidth
- $S + T = 1 \;\Rightarrow$
  $$|S(i\omega)| + |T(i\omega)| \geq 1$$
  $$\big||S(i\omega)| - |T(i\omega)|\big| \leq 1$$
- Some modes may be impossible to control or observe due to lack of controllability or observability

Limitations in control design

Fundamental limitations:

- Bode's Relation: amplitude and phase are coupled
- Bode's Integral Theorem: $|S(i\omega)|$ cannot be made small everywhere
- Limitations from unstable poles
- Limitations from right-half-plane (RHP) zeros
- Limitations from time delays

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1. Bode's Relation and Bode's Integral Theorem
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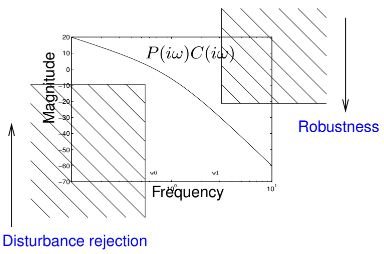
Recall: Loop shaping design

The loop transfer function $L = PC$ should be made large at low frequencies and small at high frequencies:



How quickly can we make the transition from high to low gain and still retain a good phase margin?

Bode's Relation — approximate version

If $G(s)$ is rational and stable with no RHP zeros, then

$$\arg G(i\omega) \approx \frac{\pi}{2} \frac{d \log |G(i\omega)|}{d \log \omega}$$

(Otherwise the phase is smaller – *non-minimum phase*)

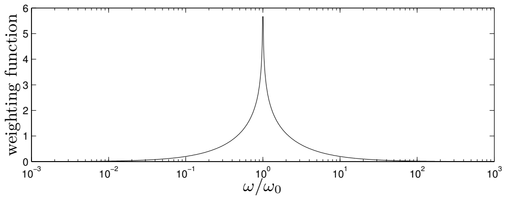
Consequence:

To have 30°—60° phase margin, the downward slope of the amplitude curve should be approximately between 1.3 and 1.7 at the crossover frequency.

Bode's Relation — exact version

If $G(s)$ is rational and stable with no RHP zeros, then

$$\arg G(i\omega_0) = \frac{2\omega_0}{\pi}\int_0^\infty \frac{\log|G(i\omega)| - \log|G(i\omega_0)|}{\omega^2-\omega_0^2}d\omega$$
$$= \frac{1}{\pi}\int_0^\infty \frac{d\log|G(i\omega)|}{d\log\omega}\underbrace{\log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|}_{\text{weighting function}}d\log\omega$$



Bode's Integral Theorem – stable case

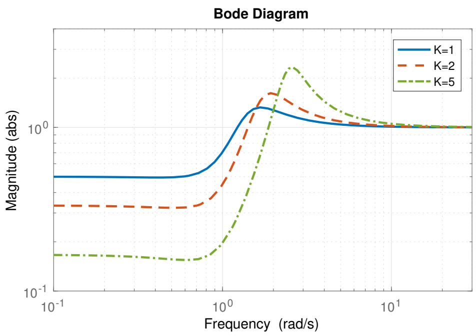
For a stable system with loop gain with relative degree ≥ 2 the following *conservation law* for the sensitivity function holds:

$$\int_0^\infty \log|S(i\omega)|d\omega = 0$$

(Sometimes known as the "waterbed effect")

Example

P-control of $(s^2+s+1)^{-1}$



Bode's Integral Theorem – general case

For a system with loop gain with relative degree ≥ 2 and unstable poles $p_1,\dots,p_M$, the following *conservation law* for the sensitivity function holds:

$$\int_0^\infty \log|S(i\omega)|d\omega = \pi\sum_{i=1}^M \text{Re}(p_i)$$

(See G&L Theorem 7.3 for details)

A similar condition relating T and RHP zeros exists, see G&L Theorem 7.5)

G. Stein: "Conservation of dirt!"

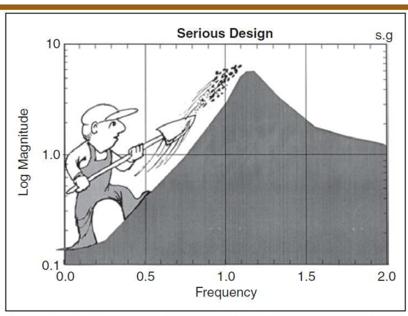


Figure 3. *Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.*

Picture from Gunter Stein's Bode Lecture (1985) "Respect the unstable". Reprint in *IEEE Control Systems Magazine*, Aug 2003.

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Unstable poles – intuitive reasoning

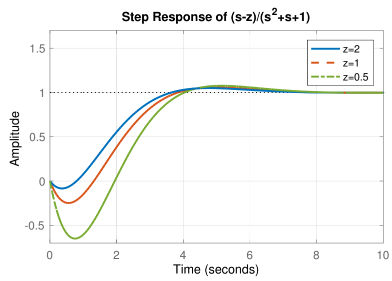
An unstable pole p makes the output signal grow exponentially as $\sim e^{pt}$ for a bounded input. To stabilize this system, one has to act fast, on a time scale $\sim 1/p$.

Conclusion: An unstable pole p gives a lower bound on the speed of the closed loop. The cross-over frequency has to fulfill

$$\omega_c \gtrsim p$$

RHP zeros – intuitive reasoning

The step response of a system with a right-half-plane zero has an undershoot. The effect is more severe if the zero is close to the origin.



Conclusion: A RHP zero z gives an upper bound on the speed of the closed loop. The cross-over frequency has to fulfill $\omega_c \lesssim z$.

Time delays – intuitive reasoning

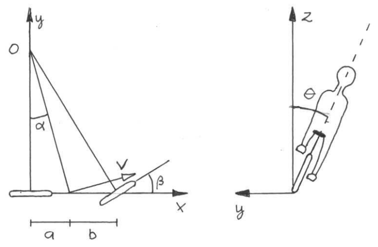
Assume that the system contains a time-delay T. This means a disturbance is not visible in the output signal until after at least T time units. This puts a hard constraint on how quickly a feedback controller can reject the disturbance!

Conclusion: A time delay T give an upper bound on the speed of the closed loop. The cross-over frequency has to fulfill

$$\omega_c \lesssim \frac{1}{T}$$

Bike example

A (linearized) torque balance can be written as



$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

Bike example, cont'd

$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

where the physical parameters have typical values as follows:

Mass:	$m = 70$ kg
Distance rear-to-center:	$a = 0.3$ m
Height over ground:	$\ell = 1.2$ m
Distance center-to-front:	$b = 0.7$ m
Moment of inertia:	$J = 120$ kgm^2
Speed:	$V_0 = 5$ ms^{-1}
Acceleration of gravity:	$g = 9.81$ ms^{-2}

The transfer function from β to θ is

$$P(s) = \frac{mV_0\ell}{b}\,\frac{as + V_0}{Js^2 - mg\ell}$$

Bike example, cont'd

The system has an unstable pole p with time-constant

$$p^{-1} = \sqrt{\frac{J}{mg\ell}} \approx 0.4 \text{ s}$$

The closed loop system must be at least as fast as this. Moreover, the transfer function has a zero z with

$$z^{-1} = -\frac{a}{V_0} \approx -\frac{0.3 \text{ m}}{V_0}$$

For the back-wheel steered bike we have the same poles but different sign of V_0 and the zero will thus be in the RHP!

An unstable pole-zero cancellation occurs for $V_0 \approx 0.75$ m/s.

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Sensitivity bounds from unstable poles/RHP zeros

The sensitivity function must be 1 at a RHP zero z:

$$P(z) = 0 \qquad \Rightarrow \qquad S(z) := \frac{1}{1 + \underbrace{P(z)}_{0}C(z)} = 1$$

Similarly, the complementary sensitivity function must be 1 at an unstable pole p:

$$P(p) = \infty \qquad \Rightarrow \qquad T(p) := \frac{P(p)C(p)}{1 + P(p)C(p)} = 1$$

The Maximum Modulus Theorem

Suppose that all poles of the rational function $G(s)$ have negative real part. Then

$$\sup_{\text{Re } s \geq 0} |G(s)| = \sup_{\omega \in \mathbf{R}} |G(i\omega)|$$

Consequences of the Maximum Modulus Theorem

Consequence for system with RHP zero z:

$$M_s = \sup_{\omega} |S(i\omega)| = \sup_{\text{Re } s \geq 0} |S(s)| \geq |S(z)| = 1$$

More interesting to use a weighting function:

$$\sup_{\omega} |W_S(i\omega)S(i\omega)| = \sup_{\text{Re } s \geq 0} |W_S(s)S(s)| \geq |W_S(z)|$$

Similar calculations can be done relating unstable poles and $T(s)$.

Consequences of the Maximum Modulus Theorem

Assume that $W_S(s)$ and $W_T(s)$ are stable transfer functions. Then we have the following **necessary** conditions:

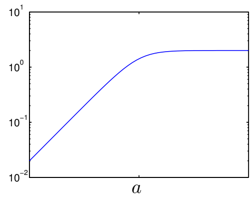
- The specification
$$\|W_S S\|_\infty \leq 1$$
is only possible to meet if $|W_S(z_i)| \leq 1$ for all RHP zeros z_i

- The specification
$$\|W_T T\|_\infty \leq 1$$
is only possible to meet if $|W_T(p_i)| \leq 1$ for all unstable poles p_i

Example: Hard limitation from RHP zero

Assume the sensitivity specification $W_S = \frac{s+a}{2s},\ a > 0$.

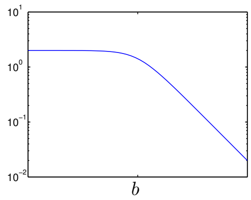
If the plant has a RHP zero in z, then the specification possible to meet only if
$$\frac{z+a}{2z} \leq 1 \quad \Leftrightarrow \quad a \leq z$$



Example: Hard limitation from unstable pole

Assume the complementary sensitivity specification $W_T = \frac{s+b}{2b}$, $b > 0$

If the plant has an unstable pole p, then the specification is possible to meet only if
$$\frac{p+b}{2b} \leq 1 \quad \Leftrightarrow \quad b \geq p$$



RHP zero and unstable pole

For a system with both RHP zero z and unstable pole p it can be shown that
$$M_s = \sup_\omega |S(i\omega)| \geq \left|\frac{z+p}{z-p}\right|$$

(See lecture notes for details)

If $p \approx z$ the sensitivity function must have a high peak *for every controller* C.

Example: Bicycle with rear wheel steering
$$\frac{\theta(s)}{\delta(s)} = \frac{am\ell V_0}{bJ} \cdot \frac{(-s+V_0/a)}{(s^2 - mg\ell/J)}$$

Lecture 7 – summary

- Bode's Relation and Bode's Integral Formula
- Limitations from unstable poles, RHP zeros and time delays
  - Intuition
  - Rules of thumb for achievable ω_c
- Limitations from unstable poles/zeros: Hard proofs using Maximum Modulus Theorem
- A back-wheel steered bicyle – pole *and* zero i RHP