FRTN10 Multivariable Control, Lecture 6

Automatic Control LTH, 2017

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

6. **Controllability/observability, multivariable poles/zeros, realizations**
7. Fundamental limitations
8. Multivariable and decentralized control

L9-L11 Controller optimization: Analytic approach

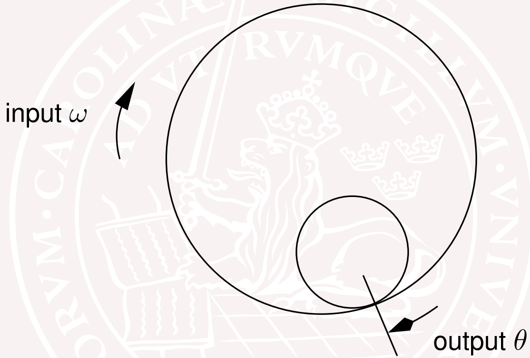
L12-L14 Controller optimization: Numerical approach

Lecture 6 – Outline

1. Controllability and observability, Gramians
2. Multivariable poles and zeros
3. Minimal realizations

[Glad & Ljung] Ch. 3.2–3.3, beg. of 3.5; Lecture notes on course web page

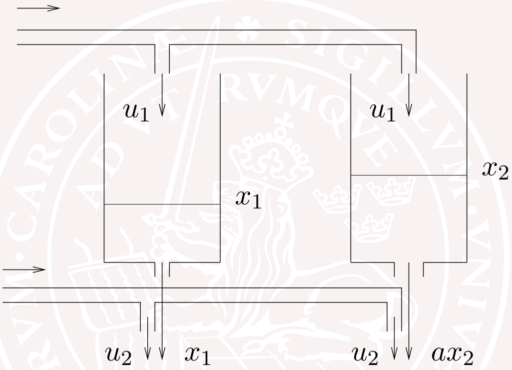
Example: Ball in the Hoop



$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$

Can you reach $\theta = \pi/4$, $\dot{\theta} = 0$? Can you stay there?

Example: Two water tanks



$a \geq 1$

$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$
$$\dot{x}_2 = -ax_2 + u_1 \qquad y_2 = ax_2 + u_2$$

Can you reach $y_1 = 1, y_2 = 2$? Can you stay there?

Review: State feedback and controllability

Process

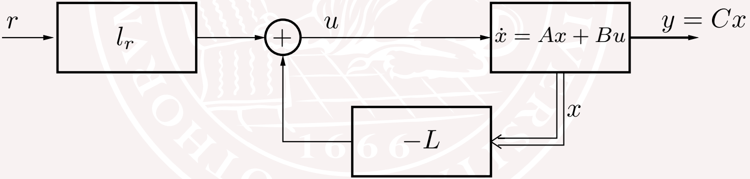
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Closed-loop system

$$\begin{cases} \dot{x} = (A - BL)x + Bl_r r \\ y = Cx \end{cases}$$

State-feedback control

$$u = -Lx + l_r r$$



If the system (A, B) is *controllable* then we can place the eigenvalues of $(A - BL)$ wherever we want

Review: State observers and observability

Process

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Observer (“Kalman filter”)

$$\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$$

Estimation/observer error $\tilde{x} = x - \hat{x}$:

$$\dot{\tilde{x}} = (A - KC)\tilde{x}$$

If the system (A, C) is *observable* then we can place the eigenvalues of $(A - KC)$ wherever we want

Controllability – definition

The system

$$\dot{x} = Ax + Bu$$

is **controllable** , if for every $x_1 \in \mathbf{R}^n$ there exists $u(t), t \in [0, t_1]$, such that $x(t_1) = x_1$ can be reached from $x(0) = 0$.

The collection of vectors x_1 that can be reached in this way is called the **controllable subspace**.

(Matlab: `orth(ctrb(A,B))`)

Controllability criteria

The following controllability criteria for a system $\dot{x} = Ax + Bu$ of order n are equivalent:

(i) rank $[B \ AB \ldots A^{n-1}B] = n$

(ii) rank $[\lambda I - A \quad B] = n$ for all $\lambda \in \mathbf{C}$

If the system is exponentially stable, define the **controllability Gramian**

$$S = \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

For such systems there is a third equivalent criterion:

(iii) The controllability Gramian is non-singular

Interpretation of the controllability Gramian

The controllability Gramian measures how difficult it is to reach a certain state.

In fact, in order to reach $x = x_1$ starting from $x = 0$ it is necessary that

$$\int_0^\infty |u(t)|^2 dt \geq x_1^T S^{-1} x_1$$

(For details, see the lecture notes.)

Computing the controllability Gramian

The controllability Gramian $S = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ can be computed by solving the Lyapunov equation

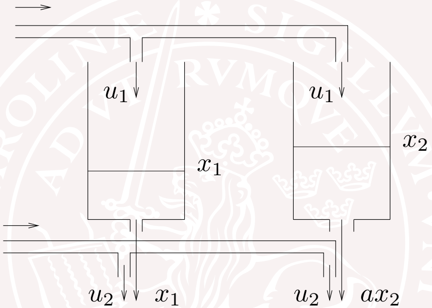
$$AS + SA^T + BB^T = 0$$

(For proof, see the lecture notes.)

Matlab: `S = lyap(A,B*B')`

Q: Where have we seen this equation before?

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad\qquad \dot{x}_2 = -ax_2 + u_1$$

Controllability Gramian: $S = \int_0^\infty \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$

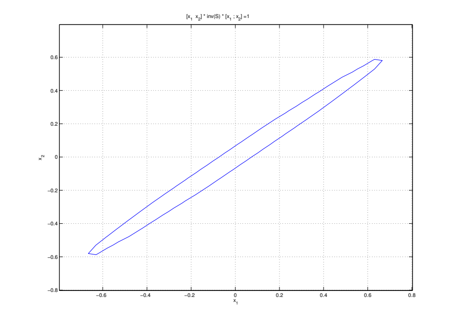
S close to singular when $a \approx 1$. Interpretation?

Example cont'd

Matlab:

```
>> a = 1.25;  A = [-1 0; 0 -1*a];  B = [1; 1];

>> Ws= [B  A*B], rank(Ws)
Ws =
    1.0000   -1.0000
    1.0000   -1.2500
ans =
     2
>> S = lyap(A,B*B')
S =
   0.5000    0.4444
   0.4444    0.4000
>> invS = inv(S)
invS =
    162.0   -180.0
   -180.0    202.5
```



Plot of $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot S^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$

corresponds to the states we can reach by $\int_0^\infty |u(t)|^2 dt = 1.$

Observability – definition

The system

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

is **observable**, if the initial state $x(0) = x_0 \in \mathbf{R}^n$ can be uniquely determined by the output $y(t), t \in [0, t_1]$.

The collection of vectors x_0 that cannot be distinguished from $x = 0$ is called the **unobservable subspace**.

(Matlab: `null(obsv(A,C))`)

Observability criteria

The following observability criteria for a system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ of order n are equivalent:

(i) rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

(ii) rank $\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbf{C}$

If the system is exponentially stable, define the **observability Gramian**

$$O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

For such systems there is a third equivalent statement:

(iii) The observability Gramian is non-singular

Interpretation of the observability Gramian

The observability Gramian measures how difficult it is to distinguish an initial state from zero by observing the output.

In fact, the influence of the initial state $x(0) = x_0$ on the output $y(t)$ satisfies

$$\int_0^\infty |y(t)|^2 dt = x_0^T O x_0$$

Computing the observability Gramian

The observability Gramian $O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ can be computed by solving the Lyapunov equation

$$A^T O + OA + C^T C = 0$$

Matlab: `O = lyap(A',C'*C)`

Mini-problem

Is the water tank system with $a = 1$ observable?

What if only y_1 is available?

Lecture 6 – Outline

1. Controllability and observability
2. Multivariable poles and zeros
3. Minimal realizations

Poles and zeros

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$Y(s) = \underbrace{\left[C(sI - A)^{-1}B + D\right]}_{G(s)} U(s)$$

For scalar systems,

- the points $p \in \mathbb{C}$ where $G(p) = \infty$ are called **poles**
- the points $z \in \mathbb{C}$ where $G(z) = 0$ are called **zeros**

Poles and zeros

For multivariable systems,

- the points $p \in \mathbb{C}$ where any $G_{ij}(p) = \infty$ are called **poles**
- the points $z \in \mathbb{C}$ where $G(z)$ loses rank are called **(transmission) zeros**

Example:

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: -2 and -1 (but what about their multiplicity?)

Zeros: 1 (but how to find them?)

Poles and zeros

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Poles: -2 and -1 (but what about their multiplicity?)

Zeros: 1 (but how to find them?)

Pole and zero polynomials

- The **pole polynomial** is the least common denominator of all minors (sub-determinants) of $G(s)$.
- The **zero polynomial** is the greatest common divisor of the maximal minors of $G(s)$, normalized to the have the pole polynomial as denominator.

The **poles** of G are the roots of the pole polynomial.

The **(transmission) zeros** of G are the roots of the zero polynomial.

Poles and zeros – example

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: Minors: $\frac{2}{s+1}, \frac{3}{s+2}, \frac{1}{s+1}, \frac{1}{s+1}, \frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{-(s-1)}{(s+1)^2(s+2)}$

The least common denominator is $(s+1)^2(s+2)$, giving the poles -2 (with multiplicity 1) and -1 (with multiplicity 2)

Zeros: Maximal minor: $\frac{-(s-1)}{(s+1)^2(s+2)}$ (already normalized)

The greatest common divisor is $s-1$, giving the (transmission) zero 1 (with multiplicity 1)

Poles and zeros – example

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: Minors: $\frac{2}{s+1}, \frac{3}{s+2}, \frac{1}{s+1}, \frac{1}{s+1}, \frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{-(s-1)}{(s+1)^2(s+2)}$

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Poles and zeros – example

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: Minors: $\frac{2}{s+1}$, $\frac{3}{s+2}$, $\frac{1}{s+1}$, $\frac{1}{s+1}$, $\frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{-(s-1)}{(s+1)^2(s+2)}$

The least common denominator is $(s+1)^2(s+2)$, giving the poles -2 (with multiplicity 1) and -1 (with multiplicity 2)

Zeros: Maximal minor: $\frac{-(s-1)}{(s+1)^2(s+2)}$ (already normalized)

The greatest common divisor is $s-1$, giving the (transmission) zero 1 (with multiplicity 1)

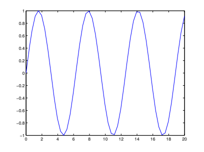
Interpretation of poles and zeros

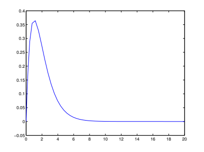
Poles:

- A pole $s = a$ is associated with the state response $x(t) = x_0 e^{at}$
- A pole $s = a$ is an eigenvalue of A

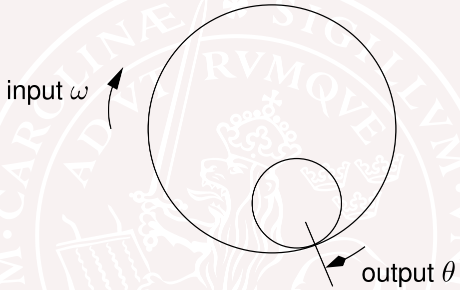
Zeros:

- A zero $s = a$ means that an input $u(t) = u_0 e^{at}$ is blocked
  - For a multivariable system, blocking occurs only in a certain input direction
- A zero describes how inputs and outputs couple to states





Example: Ball in the Hoop

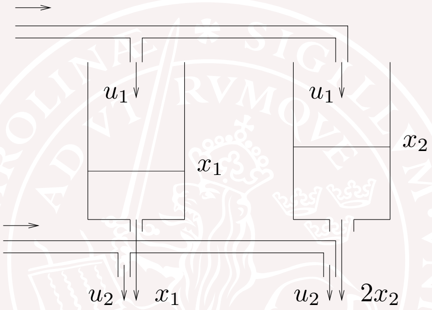


$$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$$

The transfer function from ω to θ is $\frac{s}{s^2+cs+k}$. The zero in $s = 0$ makes it impossible to control the stationary position of the ball.

- Zeros are not affected by feedback!

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$

$$\dot{x}_2 = -2x_2 + u_1 \qquad y_2 = 2x_2 + u_2$$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix} \qquad \det G(s) = \frac{-s}{(s+1)(s+2)}$$

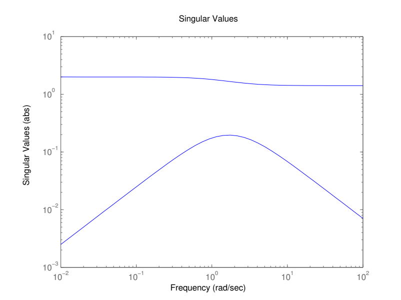
The system has a zero in the origin! At stationarity $y_1 = y_2$.

Plot singular values of $G(i\omega)$ vs frequency

```
» s=tf('s')
» G=[1/(s+1) 1 ; 2/(s+2) 1]
» sigma(G) ; plot singular values

% Alt. for a certain frequency:

» w=1;
» A = freqresp(G,i*w);
» [U,S,V] = svd(A)
```



The largest singular value of $G(i\omega) = \begin{bmatrix} \frac{1}{i\omega+1} & 1 \\ \frac{2}{i\omega+2} & 1 \end{bmatrix}$ is fairly constant.

This is due to the second input. The first input makes it possible to control the difference between the two tanks, but mainly near $\omega = 1$ where the dynamics make a difference.

Lecture 6 – Outline

1. Controllability and observability
2. Multivariable poles and zeros
3. Minimal realizations

Minimal realization – definition

Given $G(s)$, any state-space model (A, B, C, D) that is both **controllable** and **observable** and has the same input–output behavior as $G(s)$ is called a **minimal realization**.

A transfer function with n poles (counting multiplicity) has a minimal realization of order n.

Realization in diagonal form

Consider a transfer function with partial fraction expansion

$$G(s) \quad = \quad \sum_{i=1}^{n} \frac{C_i B_i}{s - p_i} + D$$

This has the realization

$$\dot{x}(t) = \begin{bmatrix} p_1 I & & 0 \\ & \ddots & \\ 0 & & p_n I \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix} x(t) + Du(t)$$

The rank of the matrix $C_i B_i$ determines the necessary number of columns in B_i and the multiplicity of the pole p_i.

(Note: Matlab has no good command for doing this. Don't use `minreal`.)

Realization of multivariable system – example 1

To find a minimal realization for the system

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

with poles in -2 and -1 (double), write the transfer matrix as (e.g.)

$$G(s) = \frac{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+2}$$

giving the realization

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix} x$$

Realization of multivariable system – example 2

To find state space-realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+3)} \\ \frac{6}{(s+2)(s+4)} & \frac{1}{s+2} \end{bmatrix}$$

write the transfer matrix as

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+3} \\ \frac{3}{s+2} - \frac{3}{s+4} & \frac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}}{s+1} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}3 & 1\end{bmatrix}}{s+2} + \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & -1\end{bmatrix}}{s+3} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}-3 & 0\end{bmatrix}}{s+4}$$

This gives the realization

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\\ \dot{x}_3(t)\\ \dot{x}_4(t)\end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix}\begin{bmatrix}u_1(t)\\ u_2(t)\end{bmatrix}$$

$$\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$