Solution to Exam in FRTN10 Multivariable Control 2017-01-03

1 a. The poles are determined by the smallest common denominator of the sub-determinants of $G(s)$. The sub-determinants are:

$$\frac{2}{(s+10)(s+1)}, \quad \frac{1}{s+1}, \quad \frac{2}{s+2}, \quad \frac{1}{s+2}, \quad \frac{-2(s+9)}{(s+10)(s+1)(s+2)},$$

where the first four are the 1×1 sub-determinants of $G(s)$ and the last one is the full 2×2 determinant. The smallest common denominator among the sub-determinants is $(s+1)(s+2)(s+10)$. The poles thus all have multiplicity 1 and are located in -1, -2 and -10.

The zeros are determined by the largest common divisor of the numerators of the largest sub-determinants, normalized with the pole polynomial in the denominator. The largest sub-determinant in this case is the full determinant of $G(s)$, and since it is already has the pole polynomial as its denominator we immediately see that the process has a zero in -9, with multiplicity 1.

b. We start by dividing $G(s)$ into separate terms using partial fraction decomposition, and then subdividing the matrix:

$$G(s) = \begin{bmatrix} \frac{-\frac{2}{9}}{s+10} + \frac{\frac{2}{9}}{s+1} & \frac{1}{s+1} \\ \frac{2}{s+2} & \frac{1}{s+2} \end{bmatrix} = \frac{1}{s+1}\begin{bmatrix} \frac{2}{9} & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix} + \frac{1}{s+10}\begin{bmatrix} -\frac{2}{9} & 0 \\ 0 & 0 \end{bmatrix}$$
$$= \frac{1}{s+1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} \frac{2}{9} & 1 \end{bmatrix} + \frac{1}{s+2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \end{bmatrix} + \frac{1}{s+10}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} -\frac{2}{9} & 0 \end{bmatrix}$$

From this form it is straightforward to obtain a diagonal state-space representation of the system:

$$\dot{x}(t) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix} x(t) + \begin{bmatrix} \frac{2}{9} & 1 \\ 2 & 1 \\ -\frac{2}{9} & 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(t)$$

c. We have:

$$G(0) = \begin{bmatrix} \frac{1}{5} & 1 \\ 1 & \frac{1}{2} \end{bmatrix} \quad G(0)^{-1} = -\frac{10}{9}\begin{bmatrix} \frac{1}{2} & -1 \\ -1 & \frac{1}{5} \end{bmatrix}$$

which gives the RGA:

$$\text{RGA}(G(0)) = G(0).*G(0)^{-T} = \begin{bmatrix} -\frac{1}{9} & \frac{10}{9} \\ \frac{10}{9} & -\frac{10}{9} \end{bmatrix}$$

Since we should avoid pairing of inputs and outputs which will result in negative diagonal elements in $\text{RGA}(G(0))$, the RGA matrix suggests that we should pair u_1–y_2 and u_2–y_1.

2. We can select e.g.

$$Q(s) = \frac{P^{-1}(s)}{(0.1s+1)^2} = \frac{0.1s+1}{s+1}$$

This gives the closed-loop system

$$Q(s)P(s) = \frac{1}{(0.1s+1)^2}$$

which has the same poles as the open-loop system. The controller is then given by

$$C(s) = \frac{Q(s)}{1-Q(s)P(s)} = \frac{\frac{0.1s+1}{s+1}}{1-\frac{1}{(0.1s+1)^2}} = \frac{0.1(s+10)^3}{s(s+1)(s+20)}$$

The controller has a pole in 0, i.e., an integrator, so the closed-loop system will be able to follow a constant reference signal without error.

3 a. A state-space realization of the process is given by

$$\dot{x} = -x + v_1$$
$$y = x + v_2$$

from which we identify $A = -1$, $N = C = 1$. The Kalman filter is given by

$$\dot{\hat{x}} = A\hat{x} + K(y - C\hat{x})$$

where $K = (PC + NR_{12})/R_2$, where $P > 0$ is given by the solution to the Riccati equation

$$2AP + R_1 - (PC + R_{12})^2/R_2 = 0$$

We obtain

$$(P+1)^2 + 2P - 6 = 0 \quad \Rightarrow \quad P = 1 \quad \Rightarrow \quad K = 2$$

Taking the Laplace transform of the Kalman filter equation and solving for $\hat{X}$ we obtain

$$\hat{X}(s) = \frac{2}{s+3} Y(s)$$

b. Let $\pi = \mathrm{E}\, x^2$. We have the Lyapunov equation

$$-1 \cdot \pi - \pi \cdot 1 + 6 = 0$$

with the solution $\pi = 3$. The variance of x is hence 3.

The spectral density of x is given by

$$\phi_x(\omega) = R_1 \frac{1}{1+i\omega} \frac{1}{1-i\omega} = \frac{6}{1+\omega^2}$$

4 a. To simplify we can look at some small parts first:

$$
\begin{aligned}
u_2 &= -C_2 y \\
y &= Fn + P_2 y_1 \\
y_1 &= \frac{P_1 C_1}{1 + P_1 C_1} u_2
\end{aligned}
$$

Putting this together we get

$$
u_2 = \frac{-(1 + P_1C_1)C_2F}{1 + P_1C_1(1 + P_2C_2)} n \Rightarrow G = \frac{-(1 + P_1C_1)C_2F}{1 + P_1C_1(1 + P_2C_2)}
$$

b. As can be seen in the step response, G is stable, and from the sigma plot we see from the maximum singular value is $\|G\|_\infty = 2$. According to the Small Gain Theorem we can then guarantee stability for all $\Delta(s)$ such that $\|\Delta\| < 1/2$. Since $\|\Delta_1\| = 0.4$, stability can be guaranteed for that one. However, $\|\Delta_2\| = 0.8$ and $\|\Delta_3\| = 1$ so stability can not be guaranteed for those.

c. Yes. It is possible that also $\Delta_2(s)$ and $\Delta_3(s)$ could result in a stable closed loop, since the Small Gain Theorem is conservative.

5 a. The dimensions of Q_1 and Q_2 give that the system has one input and two outputs.

b. The relation between the first and second output is unchanged; it's just a scaling factor. The punishment on the control signal is however relatively larger with the starred weight matrices, so the control signal will be weaker, resulting in a slower closed-loop system.

c. The state feedback vector is $L = Q_2^{-1}B^TS$, where $S = \left(\begin{smallmatrix} s_1 & s_2 \\ s_2 & s_3 \end{smallmatrix}\right)$ is the positive solution to the Riccati equation $A^TS + SA + C^TQ_1C - SBQ_2^{-1}B^TS = 0$. Inserting the system matrices we end up with the system of equations

$$
\begin{aligned}
&10 - 10s_2^2 = 0 \\
&s_1 - 10s_2s_3 = 0 \\
&2s_2 + 1 + 10s_3^2 = 0
\end{aligned}
$$

which gives

$$
S = \begin{pmatrix} \sqrt{30} & 1 \\ 1 & \sqrt{0.3} \end{pmatrix}
$$

and $L = \begin{pmatrix} \sqrt{10} & \sqrt{3} \end{pmatrix}$

d. Yes we can! Since we have full state feedback (LQ control), we are guaranteed to have a stable system with at least 60 degrees phase margin.

6 a. The missing lines should be along the lines of

```
% Constraint on transfer function n -> u
abs(Q_fr*b) <= CS_max

% Constraint on overshoot in y from reference step
PQ_sr*b <= max_overshoot;

% Constraint on control signal u for reference step
abs(Q_sr*b) <= umax;
```

b. Either of the following answers are acceptable:

1. The controllers can have very high order, which makes it computationally expensive and numerically challenging to implement them.
2. The designed controllers can be unstable which is not desirable in real-world applications.

7 a. It is easier to take closed loop robustness into account when doing loop shaping, when doing LQG design there are no robustness guarantees.

b. High control signal activity tends to wear out the actuator or make actuator nonlinearities more noticeable. The present controller has infinite high-frequency gain, implying that a low-pass filter should be added to the controller.

c. To increase the speed for which load disturbances are rejected there are a few different options: increase the integral action, add a lag filter at low frequencies, or increase the controller gain/system bandwidth. (The third option however typically also decreases the phase margin.)

d. It is not possible to conclude stability of the closed-loop system only from the magnitude plots of the Gang of Four. For example, the magnitude plots of the unstable system $1/(s-1)$ and the stable system $1/(s+1)$ are the same.

e. The design of F does not impact robustness and disturbance rejection, so it is typically best to first design the controller C for good robustness and disturbance rejection, and then design the prefilter F for a good reference step response. If F would have been design first, the design of C would affect both robustness, disturbance rejection and the reference step response, which would have made things more complicated.

f. Since the plant has a time delay, $\frac{1+PC}{PC}$ will not be causal, and this cannot be helped by increasing d. To remedy the problem, the delay must be included in F, e.g.

$$F = \frac{(1+PC)e^{-s}}{PC(1+sT_f)^d}$$