FRTN10 Exercise 13. Controller Simplification

13.1 Consider a SISO system for which the pole-zero map is given in Figure 13.1.

a. Determine the transfer function of the system. You can assume that the static gain is $G(0) = 1$.

b. By studying the pole-zero map, it is possible to get a hint that the system is a candidate for model order reduction. How?

c. Calculate a balanced realization and the Hankel singular values of the system. Perform a model reduction by eliminating the state corresponding to the smallest singular value.
Useful commands: `balreal`, `modred`.

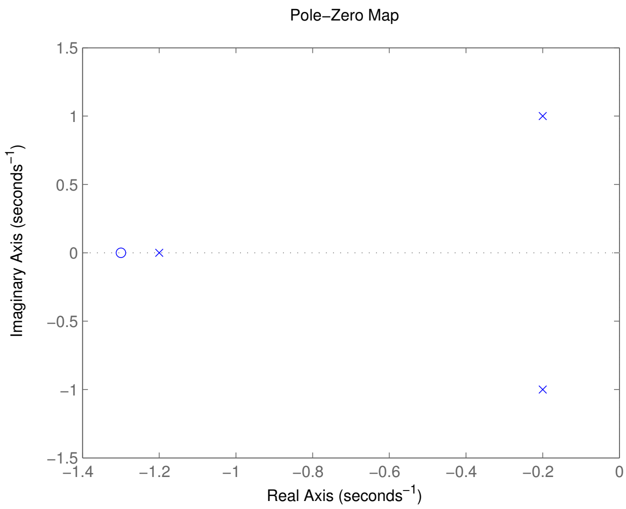

Figure 13.1 Pole-zero map of the system in Problem 13.1

13.2 For the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 10u$$

solve the following problems by hand:

a. Verify that the controllability gramian is $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ while $\begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ is the observability gramian.

b. Determine the Hankel singular values.

c. Find a coordinate change that gives a balanced realization.

d. Find a reduced system $G_1(s)$ by truncating the state corresponding to the smallest Hankel singular value.

13.3 For the same system and notation as in the previous problem, use a computer for the following:

a. Find the transfer function $G(s)$ from u to y.

b. Compare the error $\max_\omega |G(i\omega) - G_1(i\omega)|$ with the error bound for balanced truncation.

c. Find a reduced system G_2 by truncating both states and keeping just a constant gain.

d. Compare the error $\max_\omega |G(i\omega) - G_2(i\omega)|$ with the error bound for balanced truncation.

13.4 Find a reduced order approximation of

$$\frac{2s^2 + 2.99s + 1}{s(s+1)^2}$$

by writing the transfer function as the sum of an integrator and a stable transfer function, then applying balanced truncation to the stable part. You may use a computer.

Solutions to Exercise 13. Controller Simplification

13.1 a. Inspection of the locations of the poles and zeros gives us the transfer function

$$G(s) = 1.04 \frac{s/1.3 + 1}{(s/1.2 + 1)(s^2 + 0.4s + 1.04)}$$

b. The closeness of the pole-zero pair on the real axis suggests that a model reduction might be possible.

c. A balanced realization and the Hankel singular values for the system can be calculated using the Matlab command

```
>>> s = tf('s');
>>> G = 1.04*(s/1.3+1)/((s/1.2+1)*(s^2+0.4*s+1.04));
>>> [balr,g] = balreal(G);
```

which gives the following Hankel singular values:

$$g = \begin{pmatrix} 1.5105 \\ 1.0196 \\ 0.0091 \end{pmatrix}$$

Elimination of the state in the balanced realization corresponding to the smallest Hankel singular value is done in Matlab by

```
>>> modsys = modred(balr,g<0.01)
>>> modsysG = tf(modsys)
```

This gives the following transfer function for the reduced order system:

$$G_{red}(s) = 0.0181 \frac{s^2 - 2.412s + 57.49}{s^2 + 0.4086s + 1.043}$$

A Bode magnitude plot of the original system and the reduced system is shown in figure 13.1.

13.2 a. With

$$S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ -1 & -0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

we have

$$AS + SA^T + BB^T = \begin{pmatrix} -2 & 0 \\ -2 & -0.5 \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ 0 & -0.5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so S is the controllability gramian. Similarly, with

$$O = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$OA + A^T O + C^T C = \begin{pmatrix} -0.5 & 0 \\ -1 & -0.5 \end{pmatrix} + \begin{pmatrix} -0.5 & -1 \\ 0 & -0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so O is the observability gramian.

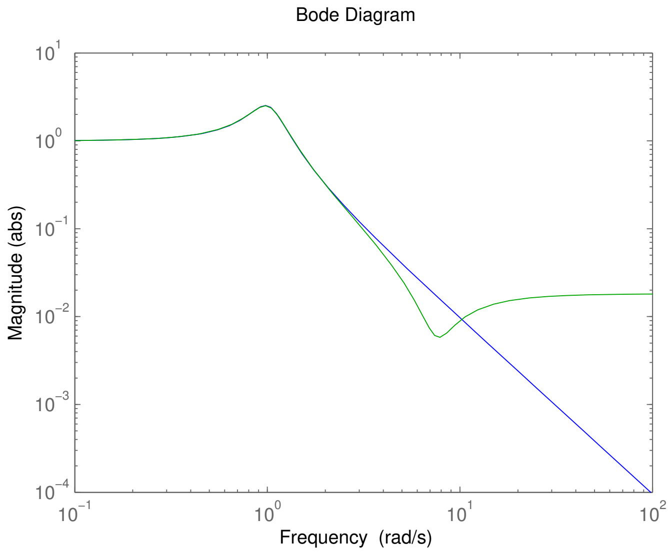

Figure 13.1 Bode magnitude plot of the original and reduced system in Problem 13.1

b. The Hankel singular values are the eigenvalues of

$$SO = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so they are both 1.

c. The coordinate change $\xi = Tx$ yields the new gramians $S_\xi = TST^T$ and $O_\xi = T^{-T}OT^{-1}$. To find T we solve the equation $S_\xi = O_\xi$. Since both S and O are diagonal it seems reasonable that a diagonal T will work. With $T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ we get the equations

$$TST^T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 2{t_1}^2 & 0 \\ 0 & {t_2}^2 \end{pmatrix}$$

and

$$T^{-T}OT^{-1} = \begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix}\begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix} = \begin{pmatrix} 0.5/{t_1}^2 & 0 \\ 0 & 1/{t_2}^2 \end{pmatrix}$$

which gives

$$\begin{aligned} 2{t_1}^2 = 0.5/{t_1}^2 \quad &\Rightarrow {t_1}^4 = 1/4 \quad \Rightarrow t_1 = 1/\sqrt{2} \\ {t_2}^2 = 1/{t_2}^2 \quad &\Rightarrow {t_2}^4 = 1 \quad \Rightarrow t_2 = 1 \end{aligned}.$$

(You could also use the direct formula for T in the proof on page 81 in [Glad & Ljung])

With this T

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix}$$

the gramians become

$$S_\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad O_\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, a balanced realization is

$$\begin{aligned}\dot{\xi} &= \hat{A}\xi + \hat{B}u \\ y &= \hat{C}\xi + \hat{D}u\end{aligned}$$

where

$$\hat{A} = TAT^{-1} = \begin{pmatrix} -1 & 0 \\ -\sqrt{2} & -0.5 \end{pmatrix} \quad \hat{B} = TB = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$
$$\hat{C} = CT^{-1} = \begin{pmatrix} \sqrt{2} & 1 \end{pmatrix} \quad \hat{D} = D$$

d. In this case, the Hankel singular values have the same size, therefore either could be removed. (However, this means that it is probably not a good idea to do any truncation at all!) If the second state is removed by letting $\dot{\xi}_2 = 0$, ξ_2 can be expressed in terms of ξ_1 through $0 = \hat{A}_{21}\xi_1 + \hat{A}_{22}\xi_2 + \hat{B}_2 u$. The reduced realization then becomes

$$\begin{aligned}\dot{\xi}_1 &= (\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})\xi_1 + (\hat{B}_1 - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{B}_2)u \\ y_r &= (\hat{C}_1 - \hat{C}_2\hat{A}_{22}^{-1}\hat{A}_{21})\xi_1 + (\hat{D} - \hat{C}_2\hat{A}_{22}^{-1}\hat{B}_2)u\end{aligned}$$

where for example $\hat{A}_{21}$ is the element in the second row and first column in $\hat{A}$.

$$\begin{aligned}\dot{\xi}_1 &= -\xi_1 + \sqrt{2}u \\ y_r &= -\sqrt{2}\xi_1 + 12u\end{aligned}$$

The transfer function is obtained through the Laplace transform

$$G_1(s) = 12 - \frac{2}{s+1}$$

13.3 a. The Matlab command `tf(ss(A,B,C,D))` gives

$$G(s) = \frac{10s^2 + 18s + 5}{s^2 + 1.5s + 0.5}$$

b. Plotting the Bode diagram for $G(s) - G_1(s)$ through the command `bodemag(G-G1)` gives 2 as the maximal error, obtained at large frequencies. The error bound, twice the sum of the truncated singular values, also gives 2. In this case the error bound is tight.

c. Truncating both states gives

$$G_2 = \hat{D} - \hat{C}\hat{A}^{-1}\hat{B} = 10$$

d. Plotting `bodemag(G-Gr)` gives 2 as the maximal error, near $\omega = 1$. The error bound $2(1+1) = 4$ is conservative.

13.4 Through partial fractions one can write

$$\frac{2s^2 + 2.99s + 1}{s(s+1)^2} = \frac{1}{s} + \frac{s + 0.99}{(s+1)^2}$$

The Matlab command
`[G3bal,sig]=balreal(tf([1 .99],[1 2 1]))` gives

$$sig = \begin{pmatrix} 0.4950 \\ 0.00001 \end{pmatrix}$$

so one state can be removed right away.
`G3red=modred(G3bal,(sig<0.1))` yields

$$\frac{-2.525 \cdot 10^{-5} s + 1}{s + 1.01} \approx \frac{1}{s + 1.01}$$

With the integrator we get the reduced system

$$\frac{1}{s} + \frac{1}{s + 1.01} = \frac{2s + 1.01}{s(s + 1.01)}$$

The commands `balreal` and `modred` can actually be used directly on systems with an integrator since they do the separation automatically.