FRTN10 Exercise 11. Youla Parametrization, Internal Model Control

11.1 Consider the control system in Figure 11.1, designed for the stable, linear, SISO system P_0.

We first want to rewrite the system in the more general form presented in Figure 11.2. In this figure: w are the external inputs to the system (e.g. disturbances and reference), z gathers all signals that we are interested in controlling, u are the control signals from C, and y contains all signals used by the controller (e.g. reference and measurements).

a. Choose

$$w = \begin{pmatrix} d \\ n \end{pmatrix}, \quad z = \begin{pmatrix} x \\ v \end{pmatrix}$$

P in Figure 11.2 consists of a number of different subsystems as

$$P = \begin{pmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{pmatrix}$$

shows. Derive P and determine P_{zw}, P_{zu}, P_{yw} and P_{yu} as transfer function matrices.

b. Call the closed-loop system (from w to z) H. Show that $H = P_{zw} + P_{zu}C(1 - P_{yu}C)^{-1}P_{yw}$. Note that we normally have a -1 in the feedback loop. Here it is assumed that this sign is part of the controller instead, hence the minus sign in $(1 - P_{yu}C)$.

c. Determine H for the system in Figure 11.1 and rewrite it in terms of the (output) sensitivity function $S = \frac{1}{1-P_{yu}C}$ and complementary sensitivity function $T = -\frac{P_{yu}C}{1-P_{yu}C}$. Use the formula in b.

d. Rewrite H using the Q parameterization $Q = C(1 - P_{yu}C)^{-1}$. Note that all the elements in H are linear in Q.

11.2 *Note: It is recommended that you solve this problem before you start on Exercise 12.*

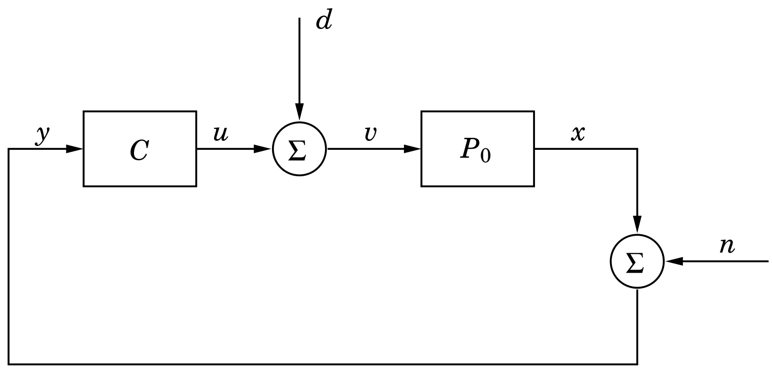

Figure 11.1 The block diagram of the closed-loop system in Problem 11.1.

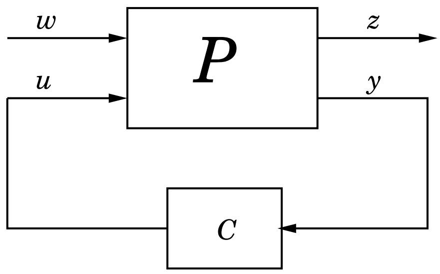

Figure 11.2 General form of a closed-loop system.

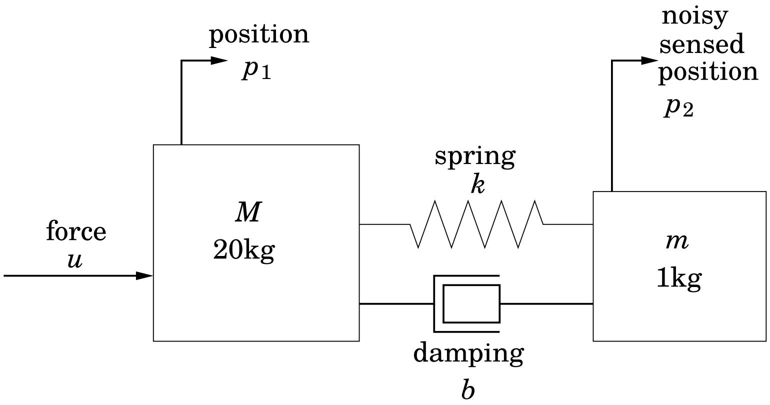

Figure 11.3 Mass spring system in Problem 11.2.

Let us consider the physical system shown in Figure 11.1, showing two masses, lightly coupled through a spring with spring constant k and damping b. The only sensor signal we have is the noisy measurement $p_2 + n$ of the position, p_2, for the small mass, m. The purpose of the controller is to make the position of the large mass, p_1, follow a reference input, r, such that the control error e becomes small. This is in turn weighted against controller effort in a quadratic cost function (the objective)

$$J = \int_0^\infty \gamma e^2(t) + \rho u^2(t) dt$$

Minimization of this function will be subject to constraints on:

- maximum magnitude of the control signal $\|u(t)\| < u_{max}$ (the force acting on the large mass) during a reference step
- step response overshoot, rise time and settling time from r to the position p_1 (performance constraint)
- the maximum norm of the sensitivity function, $\|S(i\omega)\|_\infty \leq M_s$ (robustness constraint)

The system can be described by the equations of motion

$$\begin{aligned} M\ddot{p}_1 + b(\dot{p}_1 - \dot{p}_2) + k(p_1 - p_2) &= u \\ m\ddot{p}_2 + b(\dot{p}_2 - \dot{p}_1) + k(p_2 - p_1) &= 0 \end{aligned}$$

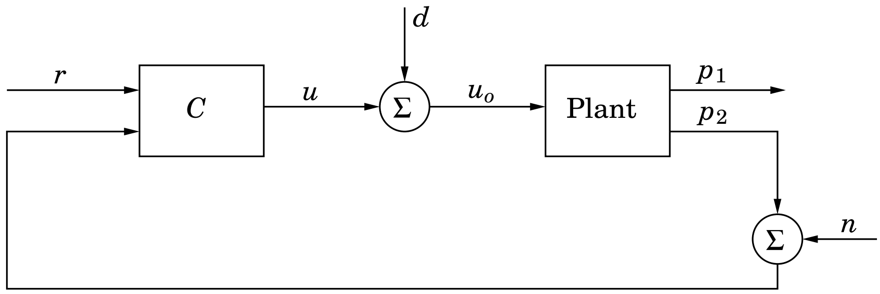

Figure 11.4 The block diagram of the closed-loop system in Problem 11.2.

Setting the plant states to

$$x = \begin{pmatrix} \dot{p}_1 \\ p_1 \\ \dot{p}_2 \\ p_2 \end{pmatrix}$$

we can rewrite the system in state-space form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ p_1 &= C_1 x \\ p_2 &= C_2 x \end{aligned}$$

where

$$\begin{aligned} A &= \begin{pmatrix} -b/M & -k/M & b/M & k/M \\ 1 & 0 & 0 & 0 \\ b/m & k/m & -b/m & -k/m \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 1/M \\ 0 \\ 0 \\ 0 \end{pmatrix} \\ C_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \\ C_2 &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

Let $M = 20$ kg, $m = 1$ kg, $k = 32$ N/m and $b = 0.3$ Ns/m.

Now, consider the problem to set up this system in a form such that we can optimize over the Q parametrization. Have Figure 11.4 as a reference. Then, the exogenous signals, w, of the system are

- the reference r,
- the noise input n,
- an input disturbance d (used for the robustness constraint).

The exogenous outputs, z, are

- the position, p_1, of the large mass M,

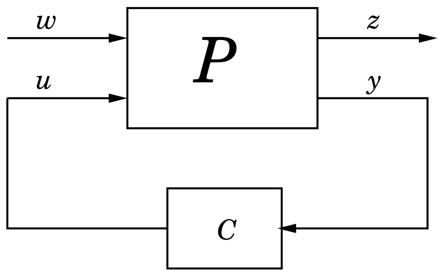

Figure 11.5 General form of a closed-loop system.

- the actuator input to the plant, u_o,
- the control error $e = r - p_1$.

The control signal, u, to the plant is

- the force u on the large mass M.

The sensed (measured) outputs y (i.e. those accessible to the controller) are

- the reference r,
- the noisy measurement $p_2 + n$.

In other words, we have

$$w = \begin{pmatrix} r \\ n \\ d \end{pmatrix}, \quad z = \begin{pmatrix} p_1 \\ u_o \\ e \end{pmatrix}, \quad y = \begin{pmatrix} r \\ p_2 + n \end{pmatrix}.$$

With these variables, we can rewrite the system in the more general form shown in Figure 11.5. In state-space form, this becomes

$$\dot{x} = Ax + B_w w + Bu \tag{11.1}$$

$$z = C_z x + D_{zw} w + D_{zu} u \tag{11.2}$$

$$y = C_y x + D_{yw} w + D_{yu} u \tag{11.3}$$

a. Determine all matrices in equations (11.1)–(11.3).

b. On the next exercise session we will use software that solves the minimization problem. This software will need to know the general process P, determined by (11.1)–(11.3), and the element indices of the closed-loop transfer function H corresponding to the constraints and cost function specified for the control design problem. For instance, the step response overshoot, rise time and settling time will correspond to $H_{p_1 r}$ which has index $(1, 1)$. Determine the rest of these indices.

c. How many inputs and outputs will the Q parametrization have?

11.3 Derive a controller using the IMC method on the following system

$$P(s) = \frac{6 - 3s}{s^2 + 5s + 6}.$$

Show that the controller has the form of a PID controller and a first order filter, i.e.

$$K\left(1+\frac{1}{T_i s}+T_d s\right)\frac{1}{sT+1}$$

11.4 Processes in industry often have time delays that give phase lags with the result of limiting the achievable performance, resulting in a *fundamental limitation*. Model based control structures that give good performance for such processes are available. Consider the simple process

$$P(s)=\frac{1}{s+1}e^{-4s},$$

which is clearly delay dominant (time delay larger than time constant).

a. Use IMC to design a delay compensating controller for this process.

b. Draw the Nyquist plot for the loop transfer function (with a chosen value of λ) and conclude whether the closed-loop system is stable.

Solutions to Exercise 11. Youla Parametrization, Internal Model Control

11.1 a. We can divide P even further into smaller parts such that

$$P_{zw} = \begin{pmatrix} P_{xd} & P_{xn} \\ P_{vd} & P_{vn} \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_{xu} \\ P_{vu} \end{pmatrix}, \quad P_{yw} = \begin{pmatrix} P_{yd} & P_{yn} \end{pmatrix}$$

Looking at the block diagram of the closed-loop system, we see that

$$\begin{aligned} &P_{xd} = P_0, \quad P_{xn} = 0, \quad P_{vd} = 1, \quad P_{vn} = 0 \\ &P_{xu} = P_0, \quad P_{vu} = 1 \\ &P_{yd} = P_0, \quad P_{yn} = 1 \\ &P_{yu} = P_0 \end{aligned}$$

Note that you have to determine the open-loop transfer functions, as if $C = 0$. The results gives us the following transfer matrix P:

$$P = \begin{pmatrix} P_0 & 0 & P_0 \\ 1 & 0 & 1 \\ P_0 & 1 & P_0 \end{pmatrix},$$

where

$$P_{zw} = \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix}, \quad P_{zu} = \begin{pmatrix} P_0 \\ 1 \end{pmatrix}, \quad P_{yw} = \begin{pmatrix} P_0 & 1 \end{pmatrix}, \quad P_{yu} = P_0$$

b.

$$\begin{aligned} u &= Cy \\ y &= P_{yu}u + P_{yw}w = P_{yu}Cy + P_{yw}w \Rightarrow y = \frac{1}{1 - P_{yw}C} P_{yw} w \\ z &= P_{zw}w + P_{zu}u = P_{zw}w + P_{zu}Cy = (P_{zw} + P_{zu}C\frac{1}{1 - P_{yw}C} P_{yw})w \end{aligned}$$

c. Using the formula, we get

$$\begin{aligned} H &= P_{zw} + P_{zu}C(1 - P_{yu}C)^{-1}P_{yw} \\ &= \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} P_0 \\ 1 \end{pmatrix} C(1 - P_0C)^{-1} \begin{pmatrix} P_0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{C}{1 - P_0C} \begin{pmatrix} P_0^2 & P_0 \\ P_0 & 1 \end{pmatrix} \\ &= \frac{1}{1 - P_0C} \begin{pmatrix} (P_0 - P_0^2C) + P_0^2C & P_0C \\ (1 - P_0C) + P_0C & C \end{pmatrix} = \begin{pmatrix} P_0S & -T \\ S & CS \end{pmatrix} \end{aligned}$$

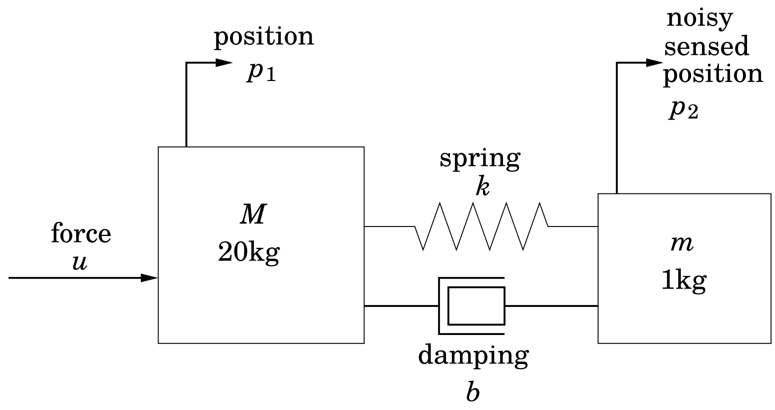

Figure 11.1 Mass spring system in Problem 11.2.

This means that the closed loop transfer function H consists of the gang of four. Note that

$$T = 1 - S = -\frac{P_0 C}{1 - P_0 C}$$

in this case where we have no explicit minus sign in the feedback loop.

d. Go back to the formula $H = P_{zw} + P_{zu}C(1 - P_{yu}C)^{-1}P_{yw}$, but replace $C(1 - P_{yu}C)^{-1}$ with Q. This gives

$$H = P_{zw} + P_{zu}QP_{yw} = \begin{pmatrix} P_0 & 0 \\ 1 & 0 \end{pmatrix} + Q \begin{pmatrix} P_0^2 & P_0 \\ P_0 & 1 \end{pmatrix} = \begin{pmatrix} P_0 + P_0^2 Q & P_0 Q \\ 1 + P_0 Q & Q \end{pmatrix}$$

where each element of H is linear in Q.

11.2 a. From the equation for the plant

$$\begin{aligned} \dot{x} &= Ax + Bu \\ p_1 &= C_1 x \\ p_2 &= C_2 x \end{aligned}$$

and the block diagram of the closed-loop system, we can see that

$$
\begin{aligned}
\dot{x} &= Ax + B(u+d) = Ax + \begin{pmatrix} 0 & 0 & B \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + Bu \\
&= Ax + B_w w + Bu \\
z &= \begin{pmatrix} p_1 \\ u_o \\ e \end{pmatrix} = \begin{pmatrix} C_1 x \\ u+d \\ r - p_1 \end{pmatrix} = \begin{pmatrix} C_1 x \\ u+d \\ r - C_1 x \end{pmatrix} \\
&= \begin{pmatrix} C_1 \\ 0 \\ -C_1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} u \\
&= C_z x + D_{zw} w + D_{zu} u \\
y &= \begin{pmatrix} r \\ p_2 + n \end{pmatrix} = \begin{pmatrix} r \\ C_2 x + n \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ C_2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} r \\ n \\ d \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} u \\
&= C_y x + D_{yw} w + D_{yu} u
\end{aligned}
$$

b. The constraint on the maximum control signal, $\|u(t)\| \leq u_{max}$, will correspond to the closed loop transfer matrix $H_{u_o r}$, with index $(2, 1)$. In problem 11.1 we saw that the transfer function $H_{u_o d}$ will correspond to the sensitivity function S. The M_s constraint will therefore correspond to the index $(2, 3)$. The objective function will be related to two indices, namely those associated with H_{er} and $H_{u_o r}$, $(3, 1)$ and $(2, 1)$.

c. We have the formula $H = P_{zw} + P_{zu} Q P_{yw}$. Since P_{zu} is a 3×1 system and P_{yw} is a 2×3 system, Q must be 1×2. Therefore we have that $Q = [Q_1 \quad Q_2]$.

11.3 The system is non-minimum phase. There are many ways to choose the Q filter for IMC, but we have to respect some fundamental limitations. Here we will use a simple choice of Q. We try to cancel the process dynamics with $Q(s)$, but use the stable counterpart $6 + 3s$ of the zero instead. We also need to add a pole to $Q(s)$ to make it proper, which we place in $s = -\lambda^{-1}$. We get

$$
Q(s) = \frac{s^2 + 5s + 6}{(6+3s)(\lambda s + 1)} = \frac{s+3}{3(\lambda s + 1)}
$$

The controller becomes

$$
C(s) = \frac{s^2 + 5s + 6}{s(3\lambda s + 6(\lambda + 1))}
$$

which can be rewritten as

$$
C(s) = \frac{5}{6(1+\lambda)} \left(1 + \frac{6}{5s} + \frac{s}{5} \right) \frac{1}{\frac{3\lambda}{6(\lambda+1)} s + 1}.
$$

This corresponds to a PID controller in series with a lowpass filter.

11.4 a. As before, there are many ways to apply IMC. Here we try the two approaches to deal with time delays described in Glad&Ljung section 8.3.

1. Choose to ignore the time delay when the $Q(s)$ transfer function is calculated, but not when $F_y(s)$ is calculated.
   Thus, we get
   $$Q(s) = \frac{(P(s)e^{4s})^{-1}}{\lambda s + 1}$$
   Hence, the controller is given by
   $$F_y(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{s+1}{\lambda s + 1 - e^{-4s}}$$

2. Approximate the time delay with a first order Padé approximation,
   $$G(s) \approx \frac{1}{s+1}\frac{1-2s}{1+2s}.$$
   When we calculate the $Q(s)$-transfer function, we exclude $1-2s$. Thus, we now have
   $$Q(s) = \frac{(s+1)(2s+1)}{(\lambda s + 1)^2}.$$
   Hence we have the controller
   $$F_y(s) = \frac{Q(s)}{1 - Q(s)P(s)} = \frac{(s+1)(2s+1)}{(\lambda s + 1)^2 - (1+2s)e^{-4s}}$$

b. The Nyquist plots can be generated in Matlab, using the following lines of code. (Note that feedback delay systems are not always handled by Control System Toolbox.) In this case, lambda is chosen to 3.

```
>> lambda = 3;
>> w = logspace(-2,2,1000);
>> P = 1./(1+i*w).*exp(-4*i*w);
>> Fy1 = (i*w+1)./(lambda*i*w+1-exp(-4*i*w));
>> Fy2 = (i*w+1).*(1+2i*w)./((lambda*i*w+1).*(lambda*i*w+1)-(1+2*i*w).*exp(-4*i*w));
>> figure
>> plot(P.*Fy1)
>> grid
>> figure
>> plot(P.*Fy2)
>> grid
```

From the plots (Figure 11.2) we see that neither encircles -1 and the closed loop systems are stable in both cases.

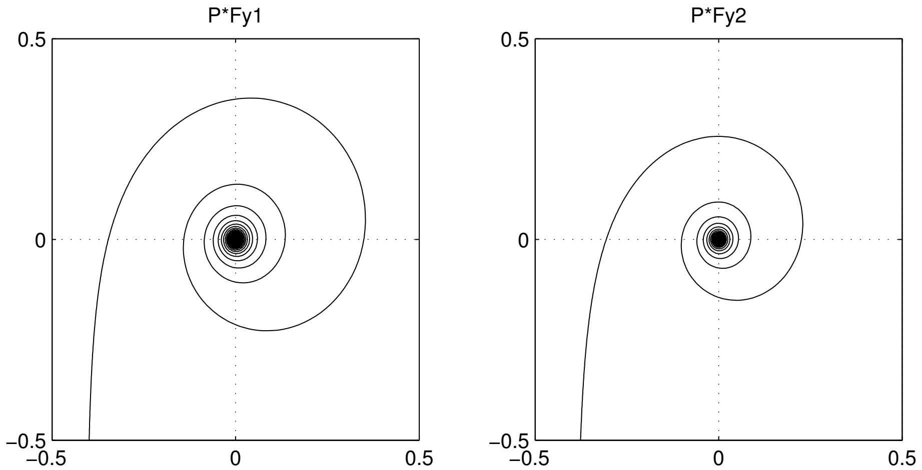

Figure 11.2 Nyquist plots of the loop transfer functions in Problem 11.4. The left plot shows the first alternative and the right plot shows the second.