FRTN10 Exercise 10. LQG, Preparations for Lab 3

10.1 Consider the system

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 6 \\ 0 & 4 \end{pmatrix} u + v_1$$
$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x + v_2$$
$$z = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

with the cost function

$$J = \mathrm{E}\left(z^2 + u^T u\right)$$

a. Design an LQG controller for the system, initially assuming that the process and measurement noise are independent and have unit intensity.

Useful commands: `lqr, kalman, lqgreg`

b. Using the states x and $\hat{x}$, write the closed-loop system in state-space form using symbols. Use L for state-feedback gain and K for Kalman filter gain.

c. Simulate the system without noise with the initial state $x = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$. Plot both process states and estimated states. The Kalman filter begins with its estimates in 0. Try different noise intensities, any conclusions?

Useful commands: `lqgreg, feedback, initial`

10.2 Consider the problem of controlling a double integrator

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + v_1$$

where the white noise v_1 has intensity I. We can only measure x_1, unfortunately with added white noise also of intensity 1. We want to minimize the cost function

$$J = \mathrm{E}\left(x_1^2 + x_2^2 + u^2\right)$$

Solve the control problem by hand and give the LQG controller in state-space form.

10.3 Do the preparatory exercises for Laboratory Session 3. The lab manual is found on the course homepage.

Solutions to Exercise 10. LQG, Preparations for Lab 3

10.1 a. See the Matlab code in **c** below.

b. Using the state vector $x_e = \begin{pmatrix} x^T & \hat{x}^T \end{pmatrix}^T$ and the obvious notation A, B, C, we get the system

$$\dot{x}_e = \begin{pmatrix} A & -BL \\ KC & A - BL - KC \end{pmatrix} x_e + \begin{pmatrix} I \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ K \end{pmatrix} v_2$$

$$z = \begin{pmatrix} C & 0 \end{pmatrix} x_e$$

c. With less measurement noise the estimated states converge faster to the actual states, and the controlled output z converges faster to zero. See Figures 10.1-10.2 and Matlab code below.

As shown in Exercise 9.1, only the relation between process noise and measurement noise matters. More process noise will therefore have the same effect as less measurement noise.

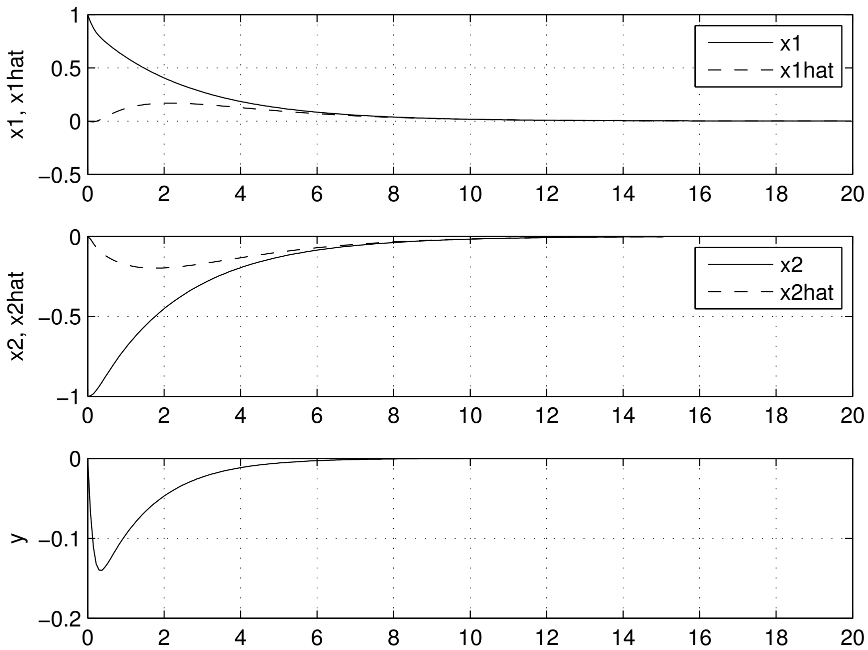

Figure 10.1 Initial response if little measurement noise ($\frac{R2}{R1} = 0.1$).

```
A = [0 1; 0 0];
B = [1 6; 0 4];
C = [1 1];

% LQ design
process = ss(A,B,C,0);
Q1 = C'*C;
Q2 = eye(2);
```

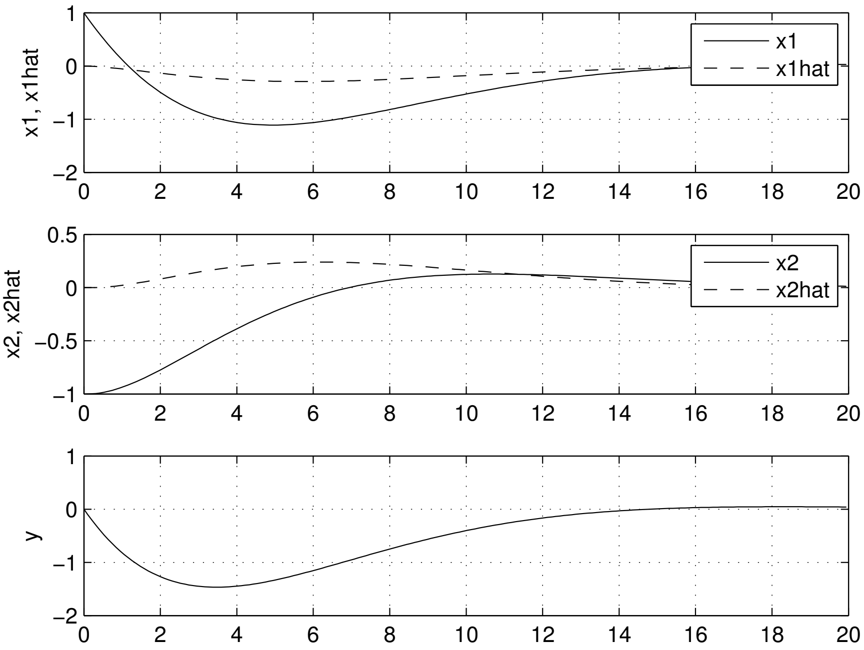

Figure 10.2 Initial response if much measurement noise ($\frac{R2}{R1} = 100$).

```
L = lqr(process,Q1,Q2);

% Kalman filter design
N = eye(2);
sysk = ss(A,[B N],C,0);
R1 = eye(2);
R2 = 1;
Kest = kalman(sysk,R1,R2);

% Construct regulator and form closed loop
reg = lqgreg(Kest,L);
closed_loop = feedback(process,-reg);

% Plot response
[Y,T,X] = initial(closed_loop,[1 -1 0 0],0:0.01:20);
subplot(311)
plot(T, X(:,1)); hold on; plot(T, X(:,3),'--'); grid
legend('x1','x1hat'); ylabel('x1, x1hat')
subplot(312)
plot(T, X(:,2)); hold on; plot(T, X(:,4),'--'); grid
legend('x2','x2hat'); ylabel('x2, x2hat')
subplot(313)
plot(T,Y); grid; ylabel('y');
```

10.2 To solve the problem we need to design an LQ state feedback and a Kalman filter. The problem parameters are given by

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad N = I,$$

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1, \quad R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R_2 = 1$$

For the LQ state feedback gain, we have to solve the Riccati equation

$$A^T S + SA + Q_1 - SBQ_2^{-1}B^T S = 0$$

with

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

This gives the following equations,

$$\begin{aligned} 1 - s_2^2 &= 0 \\ s_1 - s_2 s_3 &= 0 \\ 2s_2 + 1 - s_3^2 &= 0 \end{aligned}$$

with the solution $s_1 = s_3 = \sqrt{3}$, $s_2 = 1$. This gives the state feedback vector $L = B^T S = \begin{pmatrix} 1 & \sqrt{3} \end{pmatrix}$.

For the Kalman filter we must solve the Riccati equation

$$AP + PA^T + R_1 - PC^T CP^T = 0$$

with

$$P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$$

yielding the equations

$$\begin{aligned} 2p_2 + 1 - p_1^2 &= 0 \\ p_3 - p_1 p_2 &= 0 \\ 1 - p_2^2 &= 0 \end{aligned}$$

Reusing the solution for S we have that $p_1 = p_3 = \sqrt{3}$ and $p_2 = 1$ and $K = PC^T = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$

The LQG controller is given by

$$\begin{aligned} \dot{\hat{x}} &= (A - BL - KC)\hat{x} + Ky \\ u &= -L\hat{x} \end{aligned}$$

and we have that

$$A - BL - KC = \begin{pmatrix} -\sqrt{3} & 1 \\ -2 & -\sqrt{3} \end{pmatrix}$$

10.3 No solution provided.