FRTN10 Exercise 9. Kalman Filtering, LQG

9.1 Consider the first-order unstable system with the dynamics

$$G(s) = \frac{1}{s-1}$$

and with a state-space representation with additive noise

$$\begin{aligned} \dot{x}(t) &= x(t) + u(t) + v_1(t) \\ y(t) &= x(t) + v_2(t) \end{aligned}$$

The uncorrelated noise signals $v_i(t)$ are white with intensities R_i. We want to investigate how the optimal Kalman filter depends on noise parameters.

a. Show that the Kalman filter gain only depends on the ratio $\beta = R_1/R_2$.

b. Find the error dynamics, i.e., the dynamics of the estimation error $\tilde{x}(t) = x(t) - \hat{x}(t)$.

c. How does the error dynamics depend on the ratio $\beta = R_1/R_2$? Interpret the result for large β (process noise much larger than measurement noise), and for small β (measurement noise much larger than process noise).

9.2 A Kalman filter should be designed for the second-order system

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_1(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v_2(t) \end{aligned}$$

where v_i are uncorrelated white noise processes with intensity 1.

a. Find the optimal filter gain K using `lqe` in Matlab.

b. Find the optimal filter gain K and formulate the resulting observer using `kalman` in Matlab.

9.3 Consider the first-order stable system with dynamics

$$G(s) = \frac{1}{s+1}$$

and a state-space representation with additive noise

$$\begin{aligned} \dot{x}(t) &= -x(t) + u(t) + v_1(t) \\ y(t) &= x(t) + v_2(t) \end{aligned}$$

The noise signals $v_i(t)$ are assumed to be uncorrelated. To attenuate constant load disturbances, we would like to design an LQG controller with high low-frequency gain.

a. To model low-frequency load disturbances, we extend the plant model as

$$
\begin{aligned}
\dot{x}(t) &= -x(t) + u(t) + v_1(t) + d(t) \\
y(t) &= Cx(t) + v_2(t)
\end{aligned}
$$

where $d(t)$ is obtained by filtering white noise $v_d(t)$ through the low-pass filter

$$H(s) = \frac{1}{s + \epsilon}$$

where ϵ is a small number. Find a state-space realization of the extended system, including the noise filter, in the form

$$
\begin{aligned}
\dot{x}_e(t) &= A_e x_e(t) + B_e u(t) + N_e v_{1e}(t) \\
y &= C_e x_e(t) + v_2(t)
\end{aligned}
$$

where $v_{1e} = \begin{pmatrix} v_1 \\ v_d \end{pmatrix}$. What are the extended matrices A_e, B_e, N_e, C_e?

b. Design an LQG controller for the extended system assuming the cost function

$$J = \mathrm{E}\left(x^2 + \rho u^2\right)$$

where ρ is a small number. Assume the noise intensities $R_1 = R_d = R_2 = 1$ and pick a small number for ϵ. Verify that the resulting controller has high gain at low frequencies. (Why does ρ need to be small?)

9.4 Consider control of a DC-motor,

$$G(s) = \frac{1}{s(s+1)}$$

Introduce the state variables $x_1 = y$, $x_2 = \dot{y}$. White process noise is active on both states with intensity 1 and with input vector $\begin{pmatrix} 0.1 & 0.1 \end{pmatrix}^T$. There is also noise on the measurements with intensity 0.1. This gives the following state-space model

$$
\begin{aligned}
\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} v_1(t) \\
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v_2(t)
\end{aligned}
$$

with $R_1 = 1$, $R_2 = 0.1$ and $R_{12} = 0$

a. One wishes to use the motor to drive an external system that might be oscillatory around the frequency 0.5 rad/s, but there is no detailed knowledge about its properties. In order not to excite the oscillatory modes we would like the controller to have small gain around the this frequency. This can be achieved by extending the plant model as

$$
\begin{aligned}
\dot{x} &= Ax + Bu + Nv_1 \\
y_e &= Cx + w + v_2
\end{aligned}
$$

The extra measurement disturbance w is generated by passing white noise n through a second-order filter with a transfer function

$$H(s) = \frac{K_v s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

with $\omega_0 = 0.5$ rad/s. The parameter ζ determines the magnitude of the filter resonance peak, and we can choose e.g. $\zeta = 0.02$.

Derive the extended process model and the noise intensity matrices needed to compute the Kalman filter gain.

b. Compute the Kalman filter using `kalman` in Matlab. Plot the transfer function of the Kalman filter from y to $\hat{x}_1$ $(= \hat{y})$. Can you see the implication of the noise modeling?

c. Assuming the cost function

$$J = \mathrm{E}\left(x_1^2 + u^2\right)$$

design an LQ control law for the extended plant. Then combine the state feedback law with the Kalman filter from **b** to form a complete LQG controller. Verify that the controller also has low gain around the frequency 0.5 rad/s.

9.5 **(*)** Consider the task of estimating the states of a double integrator where noise with intensity 1 affects the input only and we have measurement noise of intensity 1.

a. Determine the optimal Kalman filter (by hand calculations).

b. What are the Kalman filter poles?

Solutions to Exercise 9. Kalman Filtering, LQG

9.1 **a.** We have $A = B = C = N = 1$. The Riccati equation thus reduces to

$$2P + R_1 - \frac{P^2}{R_2} = 0,$$

which has the positive solution $P = R_2 + R_2\sqrt{1 + \frac{R1}{R2}}$. Thus, the Kalman filter gain is

$$K = \frac{1}{R_2}P = 1 + \sqrt{1 + \frac{R_1}{R_2}} = 1 + \sqrt{1+\beta}.$$

b. The Kalman filter dynamics are given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where $y(t) = Cx(t) + v_2(t)$. Using the values $A = B = C = N = 1$ we have the error dynamics

$$\dot{\tilde{x}}(t) = (A-KC)\tilde{x}(t) - Kv_2(t) + v_1(t) = -\sqrt{1+\beta}\tilde{x}(t) - (1+\sqrt{1+\beta})v_2(t) + v_1(t)$$

c. The position of the Kalman filter pole is $-\sqrt{1+\beta}$. We can see that if $\beta \to \infty$, the pole of the Kalman filter $\to -\infty$. Hence, the estimation error dynamics are fast, and the Kalman filter very much trusts the measurements. On the other hand, if $\beta \to 0$, the Kalman filter pole tends to -1, that is, as fast as the process pole. Now, the filter trusts the model much more than the measurements.

9.2 See Matlab code below.

```
>> A = [0 1;1 0];
>> B = [1; 0];
>> C = [1 0];
>> N = [1; 1];
>> R1 = 1;
>> R2 = 1;
```

a.
```
>> % Using lqe
>> K = lqe(A,N,C,R1,R2)
K =
    2.4142
    2.4142
>>
```

b.
```
>> % Using kalman
>> sysk = ss(A,[B N],C,0);
>> [obs,K] = kalman(sysk,R1,R2)
obs =

  A =
```

```
            x1_e    x2_e
   x1_e   -2.414       1
   x2_e   -1.414       0

  B =
             u1      y1
   x1_e       1   2.414
   x2_e       0   2.414

  C =
          x1_e  x2_e
   y1_e      1     0
   x1_e      1     0
   x2_e      0     1

  D =
          u1  y1
   y1_e    0   0
   x1_e    0   0
   x2_e    0   0

Input groups:
       Name        Channels
    KnownInput        1
    Measurement       2

Output groups:
         Name         Channels
    OutputEstimate       1
    StateEstimate       2,3

Continuous-time state-space model.

K =
    2.4142
    2.4142
```

9.3 **a.** The noise model can have for instance the realization

$$\dot{d}(t) = -\epsilon\, d(t) + v_d(t)$$

Using the extended state vector $x_e = \begin{pmatrix} x \\ d \end{pmatrix}$ the extended process model is

$$\begin{aligned}
\dot{x}_e(t) &= \underbrace{\begin{pmatrix} -1 & 1 \\ 0 & -\epsilon \end{pmatrix}}_{A_e} x_e(t) + \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{B_e} u(t) + \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{N_e} v_{1e}(t) \\
y(t) &= \underbrace{\begin{pmatrix} 1 & 0 \end{pmatrix}}_{C_e} x_e(t) + v_2(t)
\end{aligned}$$

b. Matlab code:

```
epsilon = 1e-6;
rho = 1e-4;
Ae = [-1 1; 0 -epsilon];
Be = [1; 0];
Ce = [1 0];
Pe = ss(Ae,Be,Ce,0);
Ne = eye(2);

Q1e = Ce'*Ce;
Q2 = rho;
Le = lqr(Ae,Be,Q1e,Q2)

R1e = eye(2);
R2 = 1;
Ke = lqe(Ae,Ne,Ce,R1e,R2)

ctrl = reg(Pe,Le,Ke);
bode(ctrl)
```

It is seen in the Bode diagram (Figure 9.1) that the controller has high gain for low frequencies. The gain is however limited by the ρ parameter, which needs to be small to allow for large control signals that can compensate for the extra disturbance. The value of ϵ will also limit the low-frequency gain.

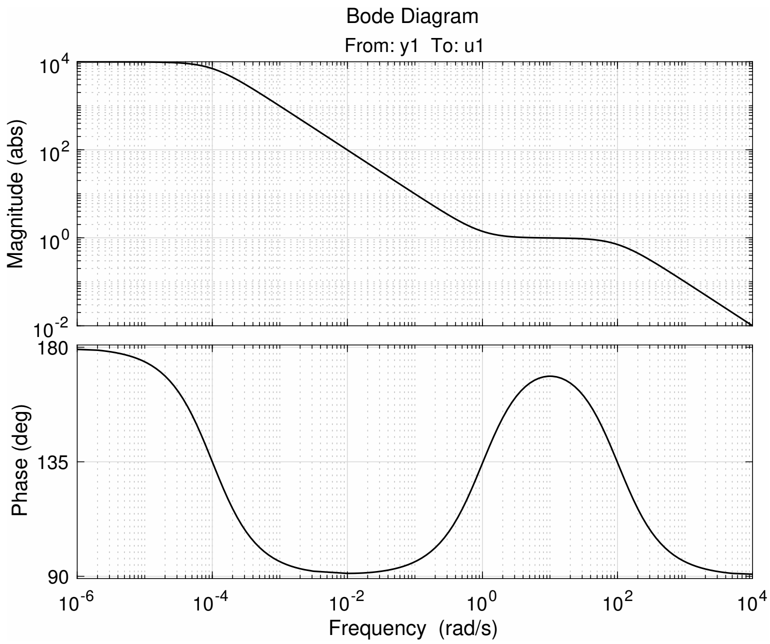

Figure 9.1 Controller Bode diagram in Problem 9.3.

9.4 **a.** In state-space form, the filter is given by (for instance)

$$\dot{x}_v(t) = \begin{pmatrix} -0.02 & -0.25 \\ 1 & 0 \end{pmatrix} x_v(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} n(t)$$

$$w(t) = \begin{pmatrix} K_v & 0 \end{pmatrix} x_v(t)$$

Extending the original state-space form with the noise model we obtain

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.02 & -0.25 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1(t) \\ n(t) \end{pmatrix}$$
$$y_e(t) = \begin{pmatrix} 1 & 0 & K_v & 0 \end{pmatrix} x(t) + v_2(t)$$

If this model is used to compute K in the Kalman filter, for an appropriate value of K_v, we get supression of the resonance frequency. The intensity of the added noise input can e.g. be set to 1 since we can control the amplitude of the disturbance by changing K_v. Thus, we have the intensity matrices $R_1 = \text{diag}(1,\ 1)$, $R_2 = 0.1$.

b. See Figure 9.2 for the Bode plot of the Kalman filter transfer function from measurement $y(t)$ to estimated process output $\hat{x}_1(t)$ using $K_v = 1$. We see a large attenuation of frequencies at $\omega = 0.5$ rad/s. (See Matlab code below.)

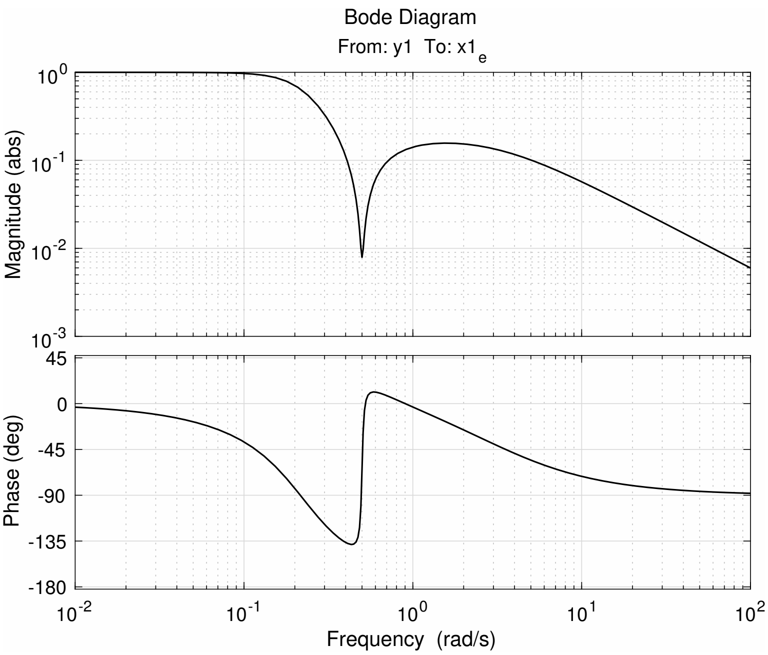

Figure 9.2 Kalman filter Bode diagram in Problem 9.4 b.

c. The Bode plot of the LQG controller transfer function from $-y$ to u is shown in Figure 9.3. Again, we see a large attenuation of frequencies around $\omega = 0.5$ rad/s.

Matlab code:

```
% Extended process model
A = [0 1 0 0; 0 -1 0 0; 0 0 -0.02 -0.2501; 0 0 1 0];
B = [0 1 0 0]'
C = [1 0 1 0];
N = [0.1 0; 0.1 0; 0 1; 0 0];
R1 = eye(2);
R2 = 0.1;
```

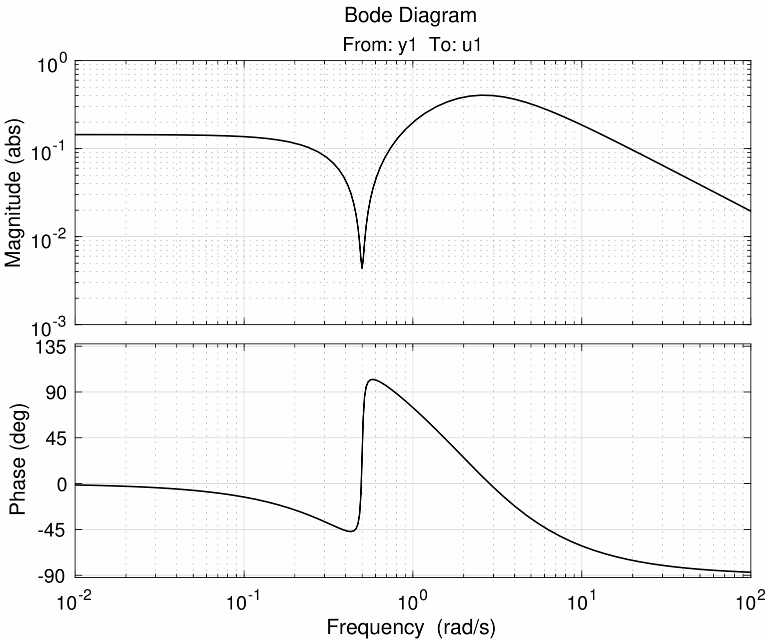

Figure 9.3 LQG controller Bode diagram in Problem 9.4 c.

```
% Design Kalman filter
sysk = ss(A,[B N],C,0);
kest = kalman(sysk,R1,R2);
Gx1hat_y = kest(2,2); % transfer function from y to x1hat
figure(1)
bode(Gx1hat_y)
grid on

% Design LQ state feedback and formulate LQG controller
Q1 = diag([1 0 0 0]);
Q2 = 0.1;
L = lqr(A,B,Q1,Q2)
P = ss(A,B,C,0);
ctrl = -lqgreg(kest,L);
figure(2)
bode(ctrl)
grid on
```

9.5 a. One possible state-space realization is

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_1(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} + v_2(t)$$

The Riccati-equation

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0$$

is solved by letting $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$. The equations become

$$\begin{aligned} 2p_2 - p_1^2 &= 0 \\ p_3 - p_1 p_2 &= 0 \\ 1 - p_2^2 &= 0 \end{aligned}$$

The positive solution is

$$P = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$$

with the optimal gain

$$K = PC^T = (\,\sqrt{2} \quad 1\,)^T$$

b. The poles of the Kalman filter are the eigenvalues of $A - KC$,

$$A - KC = \begin{pmatrix} -\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix}$$

with the eigenvalues $\lambda_j = \dfrac{1}{\sqrt{2}}(-1 \pm i)$.