FRTN10 Exercise 8. Linear-Quadratic Control

8.1 Consider the first-order unstable process

$$\dot{x}(t) = ax(t) + u(t), \qquad a > 0$$

a. Design an LQ controller $u(t) = -Lx(t)$ that minimizes the criterion

$$J = \int_0^\infty \left(x^2(t) + \rho u^2(t)\right) dt.$$

b. Do the design for different ρ using Matlab assuming $a = 1$ and plot the position of the closed-loop pole as a function of ρ. Explain how the speed of the system depends on ρ.

8.2 Consider the second-order system

$$\dot{x}(t) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t)$$
$$y(t) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(t)$$

a. Design an LQ controller $u(t) = -Lx(t)$ that minimizes the criterion

$$J = \int_0^\infty \left(y^2(t) + u^2(t)\right) dt.$$

What are the poles of the closed-loop system?

b. Solve the same problem as in **a.** by

1. using `lqr` in Matlab.
2. using `care` to solve the algebraic Riccati equation in Matlab.

Also simulate the closed-loop system from the initial condition $x(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$.

8.3 Consider a process

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u(t)$$

Show that $u(t) = -Lx(t)$ with

$$L = \begin{pmatrix} 2 & -3 \end{pmatrix}$$

can *not* be an optimal state feedback designed using linear quadratic theory with the cost function

$$J = \int_0^\infty \left(x^T(t) Q_1 x(t) + Q_2 u^2(t)\right) dt$$

where $Q_1,\ Q_2 > 0$.

Hint: Sketch the Nyquist plot of the loop transfer function $L(sI - A)^{-1}B$.

8.4 Consider the system

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} x + \begin{pmatrix} -4 \\ 8 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

One wishes to minimize the criterion

$$J(T) = \int_0^T \left(x^T(t) Q_1 x(t) + Q_2 u^2(t) \right) dt$$

Is it possible to find positive definite weights Q_1 and Q_2 such that the cost function $J(T) < \infty$ as $T \to \infty$?

8.5 We would like to control the following process with linear-quadratic optimal control:

$$\dot{x}(t) = \begin{pmatrix} 1 & 3 \\ 4 & 8 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0.1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$

The penalty on $x_1^2(t)$ should be 1, and the penalty on $x_2^2(t)$ should be 2. For $u^2(t)$ we will try different penalty values: $\rho = 0.01,\ 1,\ 100$.

a. Determine the cost function for the three different cases.

b. Assume that we want to add reference tracking so that $y = r$ in stationarity, using the control law $u(t) = L_r r(t) - Lx(t)$. In Matlab, calculate the three different resulting controllers, calculate the resulting closed-loop poles and simulate step responses from r to x_2 and from r to u. Verify that there is no static error.

8.6 Consider the double integrator

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

A set of LQ controllers $u(t) = -Lx + L_r r$ have been designed. L was calculated to minimize the cost function

$$J = \int_0^\infty \left(x^T(t) Q_1 x(t) + Q_2 u^2(t) \right) dt$$

and L_r was chosen to give unit static gain from r to y. The four plots in Figure 8.1 show the step responses of the closed-loop system for four different combinations of weights, Q_1, Q_2. Pair the combinations of weights given below with the step responses in Figure 8.1.

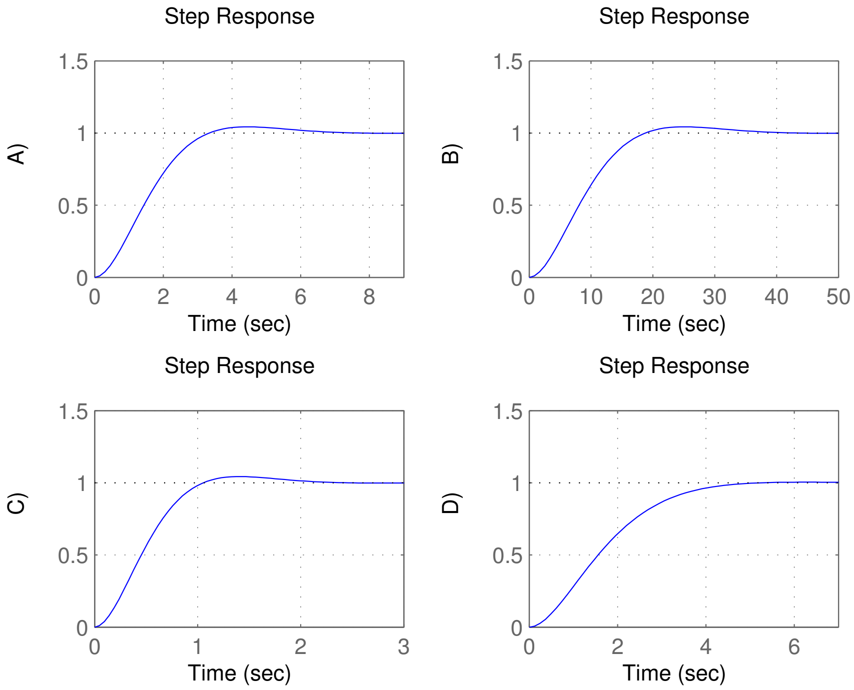

Figure 8.1 Step responses for LQ-control of the system in Problem 8.6 with different weights on Q_1, Q_2.

1)
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 0.01$$

2)
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1$$

3)
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1$$

4)
$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1000$$

8.7 **(*)** Consider the double integrator

$$\ddot{\xi}(t) = u(t).$$

with state-space representation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

where $x = (\xi(t), \dot{\xi}(t))$. You would like to design a controller using the criterion

$$\int_0^\infty (\xi^2(t) + \eta \cdot u^2(t))\, dt$$

for some $\eta > 0$.

a. Show that $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$ with

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

b. What are the closed-loop poles of the system when using this optimal state feedback? What happens with the control signal if η is reduced?

Solutions to Exercise 8. Linear-Quadratic Control

8.1 **a.** Using $A = a,\ B = 1,\ Q_1 = 1,\ Q_2 = \rho,\ Q_{12} = 0$ the Riccati equation becomes

$$2Sa + 1 - S\rho^{-1}S = 0$$

The positive solution is

$$S = a\rho + \sqrt{(a\rho)^2 + \rho}$$

and the optimal controller gain is given by

$$L = \frac{S}{\rho} = a + \sqrt{a^2 + \frac{1}{\rho}}.$$

b. See Matlab code below and Figure 8.1. Conclusion: Less weight on u gives a faster system since we are allowed to use the control signal more, and vice versa.

```
A = 1;
B = 1;
C = 1;

P = ss(A,B,C,0);

Q1 = 1;
rhovec = 0.001:0.001:0.5;
Evec = zeros(size(rhovec));

for i = 1:length(rhovec)
    rho = rhovec(i);
    [L,S,E] = lqr(P,Q1,rho);
    Evec(i) = E;
end

plot(rhovec, Evec)
xlabel('Control signal weight')
ylabel('Closed-loop pole')
grid
```

8.2 **a.** Using $y(t) = Cx(t)$ we first rewrite the cost function as

$$J = \int_0^\infty \left(x^T(t)C^TCx(t) + u^2(t)\right) dt$$

from which we identify $Q_1 = C^TC = \left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right)$, $Q_2 = 1$ and $Q_{12} = \left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\right)$.

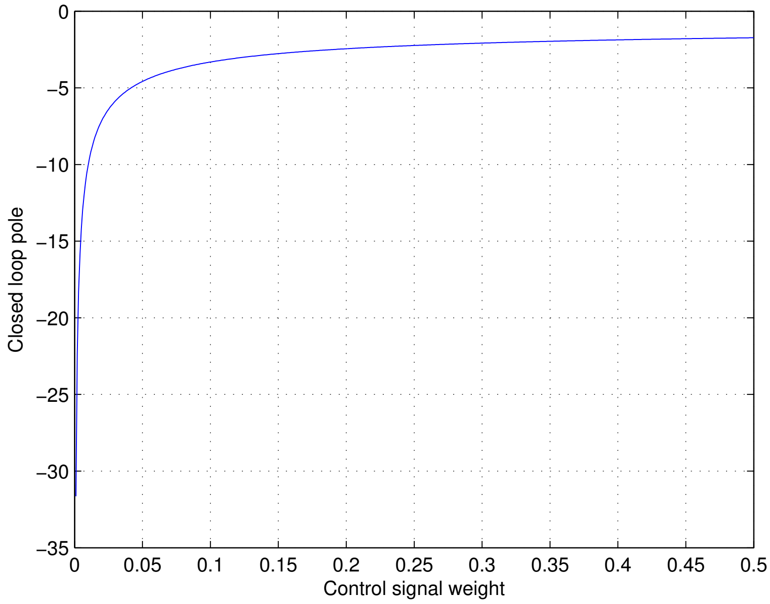

Figure 8.1 Control signal weight versus closed-loop pole

The Riccati equation becomes $Q_1 + A^T S + SA - SBB^T S^T = 0$. Let

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

We get the following system of equations:

$$\begin{aligned} 2s_1 + 2s_2 + 1 - s_1^2 &= 0 \\ s_2 + s_3 + 1 - s_1 s_2 &= 0 \\ 1 - s_2^2 &= 0 \end{aligned}$$

The solution is $s_1 = 3,\ s_2 = s_3 = 1$. This gives the state feedback vector

$$L = B^T S = (3 \quad 1).$$

The poles of the closed-loop system are given by $\det(\lambda I - A + BL) = 0$ which gives $\lambda_1 = -1$, $\lambda_2 = -1$.

b. See Figure 8.2 and Matlab code below

```
A = [1 0; 1 0];
B = [1 0]';
C = [1 1];
Q1 = C'*T;
Q2 = 1;

% 1. Using lqr
[L,S,E] = lqr(A,B,Q1,Q2)

% 2. Using care
```

```
[S,E,L] = care(A,B,Q1,Q2)

% simulate the system with initial conditions
sys = ss(A-B*L,B,C,0);
x0 = [1 1];
initial(sys,x0); grid
```

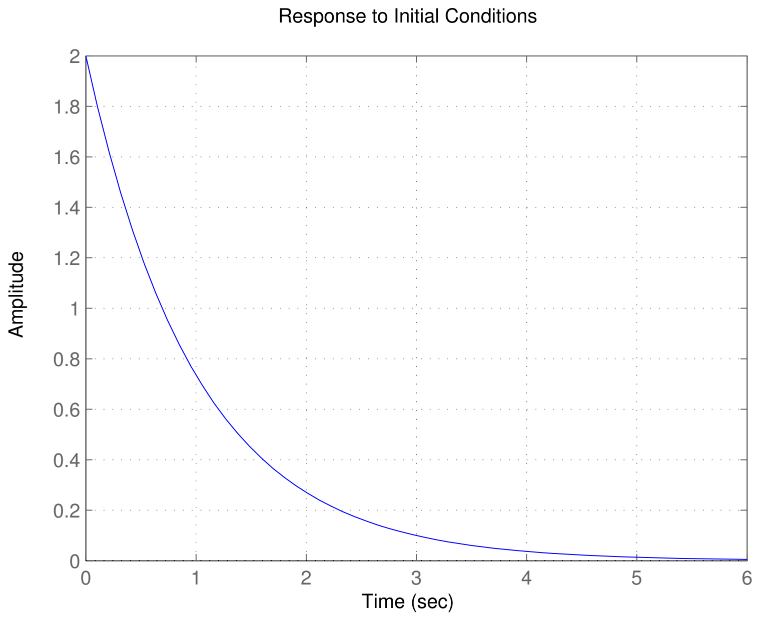

Figure 8.2 Response to initial conditions

8.3 The loop gain is

$$L(sI-A)^{-1}B = \frac{6}{(s+1)(s+2)}$$

The Nyquist curve starts on the positive real axis and will approach the origin along the negative real axis with phase -180° as $\omega \to \infty$. This is not consistent with an LQ-optimal loop gain, which will always remain at a distance ≥ 1 from the critical point -1 and will hence have an asympototic phase of -90°. Therefore, L cannot be an LQ-optimal state feedback vector.

8.4 The system has two unstable poles in 2 and 3. If the cost function should be less than ∞ then the system must be stabilizable, i.e. all unstable poles must be controllable (due to $Q_1 > 0$). The controllability matrix is given by

$$W_c = (B \quad AB) = \begin{pmatrix} -4 & -12 \\ 8 & 24 \end{pmatrix}$$

which is a rank 1 matrix. Thus, only one of the modes is controllable meaning that there is an uncontrollable, unstable mode, and hence, we can not make the cost function less than ∞.

8.5 **a.** The cost function is $J = \int_0^\infty \left(x^T(t) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x(t) + \rho u^2(t) \right) dt, \rho = 0.01,\ 10,\ 1000.$

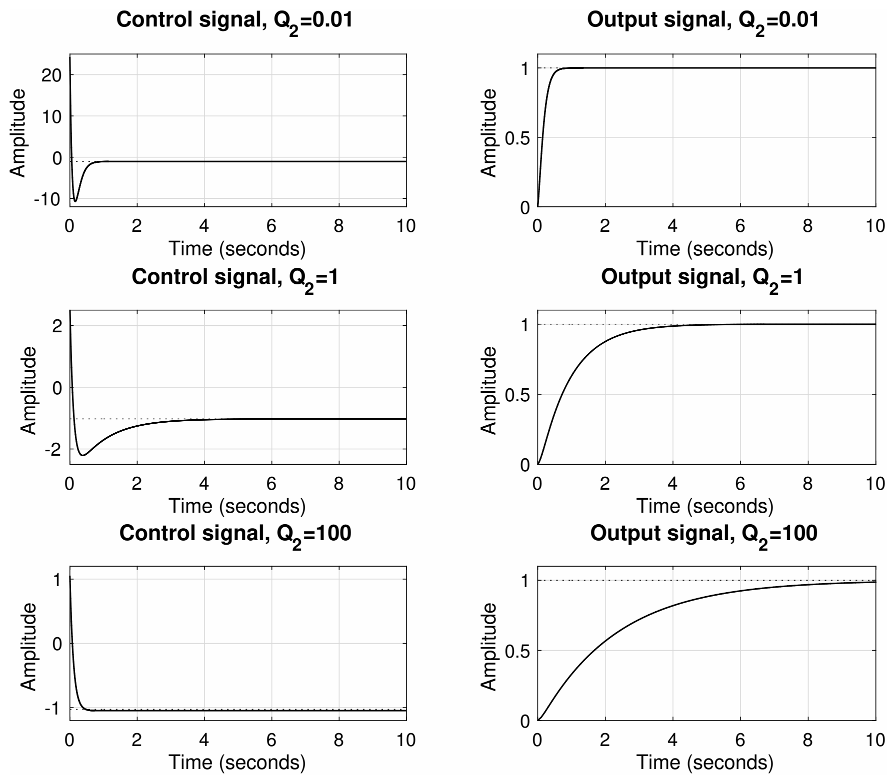

Figure 8.3 Step responses for different weight on control signal.

b. See Figure 8.3 for step responses, and Matlab code below.

```
A = [1 3; 4 8]; B = [1; 0.1]; C = [0 1];
P = ss(A,B,C,0);

Q1 = [1 0; 0 2]; Q2_vector = [0.01 1 100];

clf
for i=1:length(Q2_vector)
   [L,S,E] = lqr(P,Q1,Q2_vector(i));

   % Calculating Lr (static gain to output should be 1)
   Lr = 1/(C/(B*L-A)*B);

   % Closed loop from r to u:
   Gur = ss(A-B*L,B*Lr,-L,Lr);

   % Closed loop from r to y:
   Gyr = ss(A-B*L,B*Lr,C,0);

   % Plotting step responses
   subplot(3,2,i*2-1)
   step(Gur)
   axis([0 10 -Inf Inf])
   title(['Control signal, Q_2=' num2str(Q2_vector(i))])
   subplot(3,2,i*2)
```

```
    step(Gyr)
    axis([0 10 -Inf Inf])
    title(['Output signal, Q_2=' num2str(Q2_vector(i))])
    poles{i} = E;
end
poles{:}
```

8.6 3) is the only case with a cost on the velocity x_2. This makes the controller try to avoid rapid variations in x_1, so we get $3) - D)$, the only step response without overshoot. The weight, Q_2, on the control signal determines the speed of the system. A low weight on the control signal gives a faster system since we are allowed to use more control signal. This reveals $1) - C)$, $2) - A)$, $4) - B)$.

8.7 **a.** Weighting matrices $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $Q_2 = \eta$. The Riccati equation to be solved with respect to S is

$$A^T S + SA + Q_1 - SBQ_2^{-1}B^T S = 0$$

Put

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix},$$

which gives

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\eta} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0$$

We see, by insertion, that

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

b. The optimal state feedback is

$$\begin{aligned} L = Q_2^{-1} B^T S &= \frac{1}{\eta} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2}\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2} \cdot \eta^{-3/4} \end{pmatrix} \\ &= \frac{1}{\eta} \cdot \begin{pmatrix} \eta^{1/2} & \sqrt{2}\eta^{3/4} \end{pmatrix} = \begin{pmatrix} \eta^{-1/2} & \sqrt{2} \cdot \eta^{-1/4} \end{pmatrix} \end{aligned}$$

The poles are the eigenvalues to $A - BL$. Put $\mu = \eta^{-1/4} \Rightarrow L = \begin{pmatrix} \mu^2 & \sqrt{2} \cdot \mu \end{pmatrix}$. This gives

$$0 = \det \begin{pmatrix} s & -1 \\ \mu^2 & s + \sqrt{2} \cdot \mu \end{pmatrix} = s^2 + \sqrt{2}\mu s + \mu^2,$$

that is

$$\begin{aligned} s &= -\frac{\mu}{\sqrt{2}} \pm \sqrt{\frac{\mu^2}{2} - \mu^2} = -\frac{\mu}{\sqrt{2}} \pm i \cdot \frac{\mu}{\sqrt{2}} = \\ &= -\frac{\mu}{\sqrt{2}} \cdot (1 \pm i) = -\frac{1}{\sqrt{2} \cdot \eta^{1/4}} \cdot (1 \pm i) \end{aligned}$$

If η is reduced, the distance between the poles and the origin will increase. This means that the size of $u(t)$ will increase.