FRTN10 Exercise 5. Multivariable Zeros, Singular Values and Controllability/Observability

5.1 Consider the following system:

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} x$$

a. Show that the system is neither controllable nor observable. Also determine the uncontrollable and unobservable modes. (If you are not familiar with the concept of modes, look it up in the textbook (Glad&Ljung)).

b. Determine the transfer function of the system and the order of a minimal state-space realisation. How can this be related to the controllable and observable states of the system?

5.2 The following model of a heat exchanger was presented in the course book (see Example 2.2):

$$\dot{x} = \begin{pmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{pmatrix} x + \begin{pmatrix} 0.01 & 0 \\ 0 & 0.01 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x,$$

Here the first state represents the temperature of the cold water and the second state is the temperature of the warm water.

a. Use Matlab to calculate the controllability Gramian.

b. What state direction is the hardest to control?

5.3 (*) In the first exercise session we were given a rough model of the pitch dynamics of JAS 39 Gripen:

$$\dot{x} = \begin{pmatrix} -1 & 1 & 0 & -1/2 & 0 \\ 4 & -1 & 0 & -25 & 8 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & -20 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 3/2 & 1/2 \\ 0 & 0 \\ 20 & 0 \\ 0 & 20 \end{pmatrix} u.$$

Using Matlab:

a. Show that there is no scalar output signal that makes the system observable.

Hint: Use symbolic toolbox to determine a general C matrix and calculate the observability matrix. For instance, the following lines of Matlab code may help you:

```
>> syms c1 c2 c3 c4 c5
>> C = [c1 c2 c3 c4 c5]
>> Wo = ...
```

b. Let the output be

$$y(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} x(t).$$

Which are the non-observable modes?

5.4 In Figure 5.1 and Figure 5.2 you see two different interconnections of the two systems

$$P_1 = \frac{s+3}{s+2}, \quad P_2 = \frac{s+1}{(s+3)(s+4)(s-2)}$$

One can notice that after multiplying the two systems we can cancel a pole and a zero in $p_0 = -3$. Usually it means that the whole system is not observable, or not controllable. Which of these two situations are depicted in the systems A and B in Figure 5.1 and Figure 5.2 respectively?

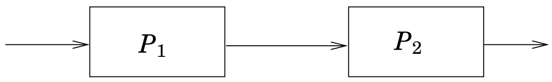

Figure 5.1 Block diagram for system A in problem 5.4.

Figure 5.2 Block diagram for system B in problem 5.4.

5.5 Consider the following transfer function matrix

$$G(s) = \begin{pmatrix} \dfrac{1}{s+2} & -\dfrac{1}{s+2} \\ \dfrac{1}{s+2} & \dfrac{s+1}{s+2} \end{pmatrix}$$

a. Determine the pole and zero polynomials for this system. What is the least order needed to realize the system in state-space form?

b. Find a state-space realization of the system.

c. Use Matlab to draw a singular value plot for the system. What is the $\mathcal{L}_2$-gain of the system? (Hint: `help sigma`)

5.6 Consider the system

$$G(s) = \begin{pmatrix} 1 & 1/s \end{pmatrix}$$

with two inputs and one output.

a. Use Matlab to determine the singular values of the system at $\omega = 1$ rad/s, together with the input directions giving the maximum and minimum output gains respectively.

b. The derived input directions are complex. What does this mean? Explain why it is logical that these input directions should give the smallest and highest system gains respectively for this particular system.

5.7 The following is an idealized dynamic model of a distillation column:

$$G(s) = \frac{1}{75s+1}\begin{pmatrix} 87.8 & -86.4 \\ 108.2 & -109.6 \end{pmatrix}$$

a. Using Matlab, plot the singular values of the process.

b. For the frequencies $\omega = 0,\ 0.1$ rad/s, calculate the gains of the system in the input directions $d_1 = [0.6713 \quad 0.7412]^T$ and $d_2 = [1 \quad 0]^T$, i.e. the amplification of the input $d_i \cdot \sin(\omega t)$ by the transfer matrix $G(s)$.

c. Determine the minimum and maximum output gains respectively at $\omega = 0$ rad/s as well as the input directions associated with them. Will the directions depend on frequency for this particular system? Explain your answer.

Solutions to Exercise 5. Multivariable Zeros, Singular Values and Controllability/Observability

5.1 **a.** In Matlab, we may derive the controllability- and observability matrices using

```
>> Wc = ctrb(A,B)

Wc =

     1    -1     1
     1    -2     4
     0     0     0

>> rank(Wc)

ans =

     2

>> Wo = obsv(A,C)

Wo =

     1     0     1
    -1     0    -3
     1     0     9

>> rank(Wo)

ans =

     2
```

Since the system is in diagonal form we can see, using Theorem 3.1 in the course book (Glad&Ljung), that the uncontrollable mode corresponds to the third state (as that row in the B matrix is 0). By Theorem 3.2 in the course book, the unobservable mode is determined to be the second state in a similar fashion (the column of C equal to 0).

The system is illustrated in the block diagram in Figure 5.1. We can see that the state x_2 will not influence y, and is therefore not observable. We can also see that the control signal u will not affect the state x_3, and therefore this state is not controllable.

b. The transfer function is simply

$$G(s) = C(sI - A)^{-1}B = \frac{1}{s+1}$$

and the system can thus be represented as a minimal realization in state-space form of order 1. Note that this corresponds to the first subsystem in Figure 5.1 which is both observable and controllable.

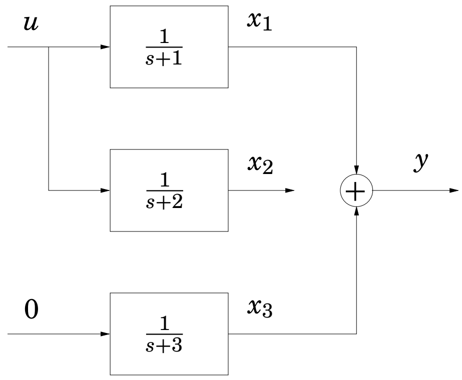

Figure 5.1

If we are only interested in the relationship between u and y, we can use the resulting first-order transfer function $G(s)$. However, the original third-order state-space model contains additional information, as seen in Figure 5.1. The second and third subsystems in this model may represent physical entities of the plant that must be taken into account. If we need to influence x_3 or monitor x_2, additional sensors or actuators are needed.

5.2 **a.** First of all, define the system in Matlab

```
>> A = [-0.21 0.2;0.2 -0.21];
>> B = 0.01*eye(2);
>> C = eye(2);
>> D = 0;
>> sys = ss(A,B,C,D);
```

The controllability Gramian is calculated using

```
>> W = gram(sys,'c')

W =

    0.0026    0.0024
    0.0024    0.0026
```

b. Recall the formula from the lecture notes:

$$\int_0^\infty |u(t)|^2 dt \geq x_1^T W^{-1} x_1$$

showing how much control effort it takes to reach the state x_1.

To identify the hardest to control state direction, we calculate the eigenvalues and the eigenvectors of W^{-1} :

```
[T L] = eig(inv(W))

T =

   -0.7071   -0.7071
```

```
   -0.7071    0.7071

L =

   1.0e+03 *

    0.2000         0
         0    8.2000
```

Apparently one eigenvalue of the inverse of the Gramian is almost 40 times larger than the other. Hence one state direction is poorly controllable.

Inspection of the corresponding eigenvectors, i.e. the columns of T, shows that the small eigenvalue correponds to a state direction where both temperatures move in the same way, while the poorly controllable state direction correponds to temperatures moving in opposite directions.

5.3 **a.** Continuing the code we get

```
>> syms c1 c2 c3 c4 c5
>> C = [c1 c2 c3 c4 c5];
>> Wo = [C;C*A;C*A^2;C*A^3;C*A^4];
>> det(Wo)

ans =

0

>> rank(Wo)

ans =

4
```

Since the system does not have full rank (5) we see that no matter how we choose C (when it is a vector), the system can never be made observable. This means that we need information from more than just one signal to make the system observable.

b. Determine the eigenvectors of the system

```
>> [V,D]=eig(A)

V =

         0    0.3333    0.4286   -0.0261    0.0206
         0    0.6667   -0.8571    0.7973   -0.3916
    1.0000    0.6667    0.2857   -0.0399    0.0196
         0         0         0    0.6017         0
         0         0         0         0    0.9197
...
```

Rewrite the system in diagonal form using the change of variables $x(t) = Vz(t)$

$$\begin{aligned}
\dot{x}(t) &= V\dot{z}(t) = AVz(t) + Bu(t) \Rightarrow \\
\dot{z}(t) &= V^{-1}AVz(t) + V^{-1}Bu(t) = \Lambda z(t) + V^{-1}Bu(t) \\
y(t) &= CVz(t)
\end{aligned}$$

where Λ is a diagonal matrix with the eigenvalues on the diagonal. Now that we have the system in the wanted form, we can determine if there are any columns in CV that are zero.

```
>> C*V

ans =

         0    0.3333    0.4286   -0.0261    0.0206
         0    0.6667   -0.8571    0.7973   -0.3916
```

The first state in z therefore corresponds to the unobservable mode. In the original variables this is the third state:

```
>> V*[1;0;0;0;0]

ans =

     0
     0
     1
     0
     0
```

So, the third state is the unobservable mode.

5.4 System A depicts the observable system. Obviously the problem is in the pole $p_0 = -3$. We control directly the plant P_1, and we observe the output of plant P_2. It means that we observe the effect of the pole $p_0 = -3$, but due to pole-zero cancellation, we cannot control it.

Similarly for system B, we control the plant P_2, and the pole $p_0 = -3$ is controllable, but the effect of that pole is cancelled by the zero in P_1 and we do not observe it. Hence the whole system is not observable.

5.5 a. The largest subdeterminant of the transfer function matrix is

$$\frac{(s+1)}{(s+2)^2} + \frac{1}{(s+2)^2} = \frac{1}{(s+2)}$$

Furthermore, the matrix elements in themselves are subdeterminants. The pole polynomial, i.e. the least common denominator of all subdeterminants, is then

$$p(s) = (s+2)$$

This means that the system has a pole in $s = -2$. The system can thus be realized in state-space form of order 1.

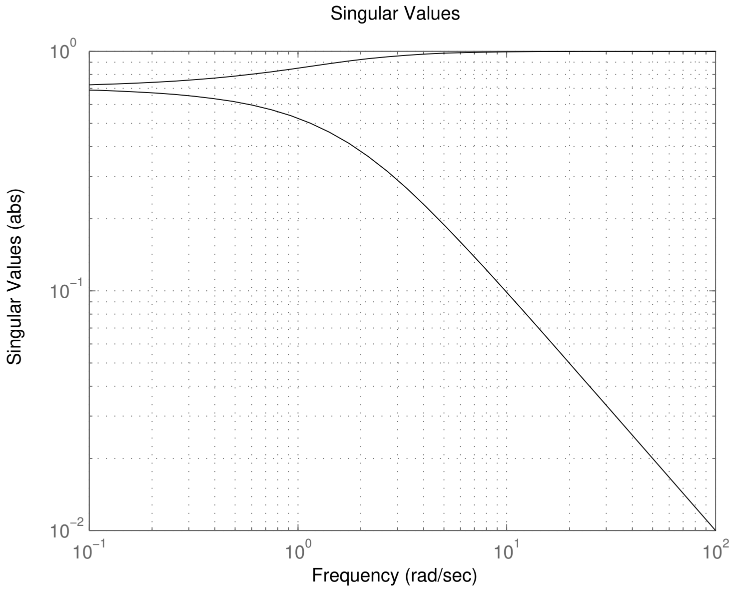

Figure 5.2 Singular value plot in Problem 5.5.

The largest possible subdeterminant was

$$\frac{1}{(s+2)}$$

The zero polynomial is thus just a constant and therefore the system does not have any zeros.

b.

$$G(s) = \begin{pmatrix} \frac{1}{s+2} & -\frac{1}{s+2} \\ \frac{1}{s+2} & \frac{s+1}{s+2} \end{pmatrix} = \begin{pmatrix} \frac{1}{s+2} & -\frac{1}{s+2} \\ \frac{1}{s+2} & 1-\frac{1}{s+2} \end{pmatrix} = \frac{1}{s+2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \frac{1}{s+2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

A state-space realization can now be written as

$$\frac{dx}{dt} = -2x + \begin{pmatrix} 1 & -1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 \\ 1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} u$$

c. The singular value plot (see Figure 5.2) is drawn using the command `sigma`. The $\mathcal{L}_2$-induced gain $||G||_\infty$ is the largest singular value of $G(i\omega)$ across all frequencies ω, from the figure we can see that $||G||_\infty = 1$ in this case. We also see that the largest gain of this system is achieved at high frequencies.

5.6 a. To determine the frequency response at a certain frequency ω, it's handy to use the Matlab command `freqresp`. To calculate the singular values together with the U and V matrices, use the function svd. The Matlab code can look like this:

```
>> s = tf('s');
>> G = [1 1/s];
>> [U,S,V] = svd(freqresp(G,1))

U =

     1


S =

    1.4142         0


V =

   0.7071          0 + 0.7071i
   0 + 0.7071i   0.7071
```

The maximum gain, corresponding to the highest singular value, is obtained as the first element in S and is $\overline{\sigma} = 1.4142$. The first column of V, $v_1 = (0.7071 \quad 0.7071i)^T$, corresponds to the input direction that gives the maximum gain $\overline{\sigma}$. Since the system has two inputs and only one output, there will always be an input direction that gives zero output (where the inputs cancel each other). The second column of V gives us this direction, $v_2 = (0.7071i \quad 07071)^T$.

b. If the input signal is a sinusoid with frequency $\omega = 1$ rad/s, the complex numbers will correspond to a phase shift of this sinusoid. The input direction giving the highest gain is $v_1 = [0.7071 \quad 0.7071i]^T$, meaning that the second input has 90° phase lead compared to the the first input.

The first input comes through the system unchanged; the second goes through an integrator, causing a phase lag of 90°. Thus the input direction $v_1 = [0.7071 \quad 0.7071i]^T$ will cause the two sinusoids that sum up at the output to be in phase; resulting in maximal gain.

If we instead use the lowest gain input direction $v_2 = [0.7071i \quad 0.7071]^T$, the second input will have a phase lag of 90°, causing a 180° phase lag at the output. The two signals will cancel at the output, resulting in minimal gain (zero).

5.7 a.
```
>> s = tf('s');
>> G = 1/(75*s+1)*[87.8 -86.4;108.2 -109.6];
>> sigma(G)
>> grid
```

See Figure 5.3 for the Matlab plot.

b. Calculate the frequency response at the given frequencies

```
>> Gfr1 = freqresp(G,0)

Gfr1 =
```

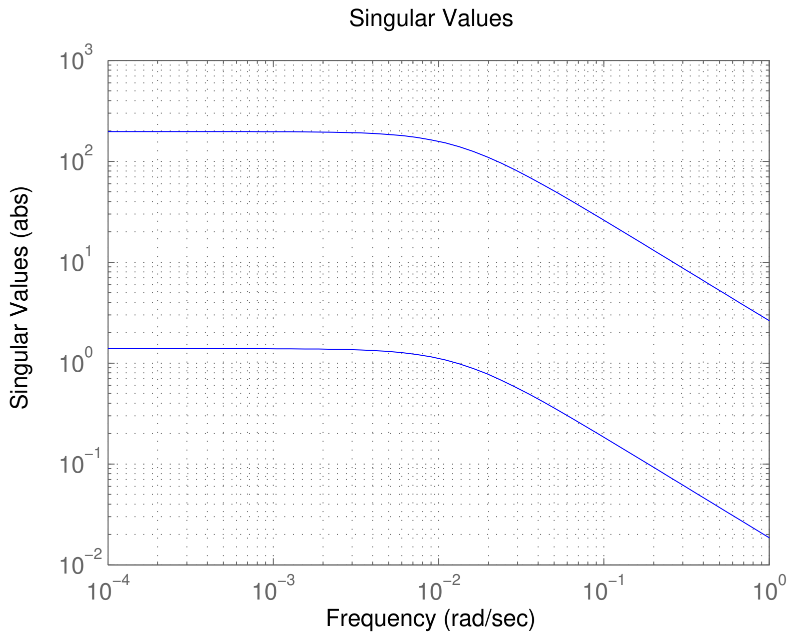

Figure 5.3 Singular value plot for Problem 5.7

```
   87.8000  -86.4000
  108.2000 -109.6000

>> Gfr2 = freqresp(G,0.1)

Gfr2 =

   1.5336-11.5022i  -1.5092+11.3188i
   1.8900-14.1747i  -1.9144+14.3581i
```

The gain of a transfer matrix at a particular frequency ω is computed as $\bar{\sigma}(G(i\omega)) = \sup_{d \neq 0} \dfrac{|G(i\omega)d|}{|d|}$. If we choose a particular input direction d_0 then the supremum disappears and the gain is given by $\dfrac{|G(i\omega)d_0|}{|d_0|}$.

Thus the gains are given by

$$\frac{|G(0)d_1|}{|d_1|} = \frac{\sqrt{(-5.1)^2 + (-8.6)^2}}{\sqrt{(0.6713)^2 + (0.7412)^2}} = \frac{10.0}{1}$$

$$\frac{|G(0)d_2|}{|d_2|} = 139.3$$

$$\frac{|G(0.1i)d_1|}{|d_1|} = 1.3$$

$$\frac{|G(0.1i)d_2|}{|d_2|} = 18.4$$

They can also be calculated in Matlab using

```
>> d1 = [0.6713;0.7412];
```

```
>> d2 = [1;0];
>> norm(Gfr1*d1),norm(Gfr1*d2),norm(Gfr2*d1),norm(Gfr2*d2)

ans =

    9.9990

ans =

  139.3416

ans =

    1.3215

ans =

   18.4159
```

c. Using Matlab:

```
>> [U,S,V] = svd(Gfr1)

U =

   -0.6246   -0.7809
   -0.7809    0.6246

S =

  197.2087         0
         0    1.3914

V =

   -0.7066   -0.7077
    0.7077   -0.7066
```

The maximum gain is $\overline{\sigma} = 197.2$ and the minimum gain is $\underline{\sigma} = 1.39$. The input direction associated with the maximum gain is $v_1 = [-0.7066 \quad 0.7077]^T$. The input direction giving the least gain is $v_2 = [-0.7077 \quad -0.7066]^T$. These directions are constant for all frequencies. The reason is that the denominators of all matrix elements are the same, which gives

$$G(i\omega) = \frac{1}{75i\omega + 1} G(0).$$

Let $G(0) = U\Sigma V^*$. We then have $G(i\omega) = U\left(\frac{1}{75i\omega+1}\Sigma\right)V^*$, and we can see that ω will only change the singular value matrix Σ, not the direction matrices U and V.