# FRTN10 Exercise 3. Disturbance Models and Robustness

**3.1** **a.** Analyze the stability of the system to the left in Figure 3.1 using the Small Gain Theorem. For what values of the gain $\|\Delta\|$ can stability be guaranteed?

**b.** Analyze the stability of the system to the right in Figure 3.1, where $K$ is a constant feedback gain. For what values of the gain $K$ can stability be guaranteed? If you get different results from **a**, explain why!

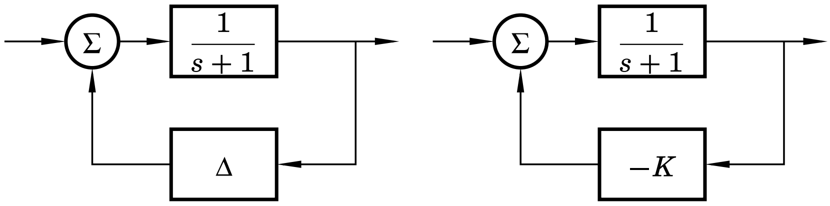



**Figure 3.1** Systems in Problem 3.1.

**3.2** Consider the system in Figure 3.2.

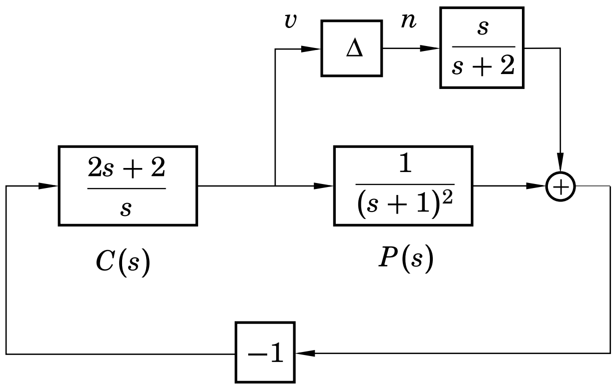



**Figure 3.2** System for Problem 3.2.

**a.** Find the transfer function $G_{vn}(s)$ from $n$ to $v$.

**b.** How large is the gain $\|G_{vn}\|$? Support the solution by a Matlab plot.

**c.** Using the Small Gain Theorem, find the largest possible gain of $\Delta$ for which the closed-loop system is stable.

**d.** The $\Delta$ block is used to account for uncertainty in the process model. Explain the role of the factor $\frac{s}{s+2}$ multiplying $\Delta$.

**3.3** A feedback system is shown in Figure 3.3.

**a.** Compute the poles of the closed-loop system.

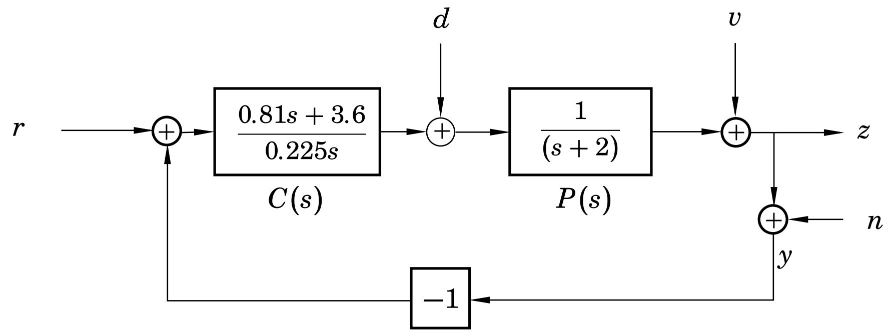



**Figure 3.3** System in Problem 3.3.

**b.** Derive the transfer functions from $r$, $d$, $v$ and $n$ to $z$. Identify the sensitivity function $S$ and the complementary sensitivity function $T$. Plot them in the same Bode plot.

**c.** If we have a disturbance $v = \sin(0.5t)$ acting on the system and all other input signals are zero, what amplitude will the oscillation in the output signal have when the transient has disappeared?

**d.** If we have a sinusoidal measurement disturbance $n(t)$ with a frequency of 50 Hz and unit amplitude and all other input signals are zero, what amplitude will the oscillation in the output signal have when the transient has disappeared?

**3.4** **(*)** Consider the system in Figure 3.4.

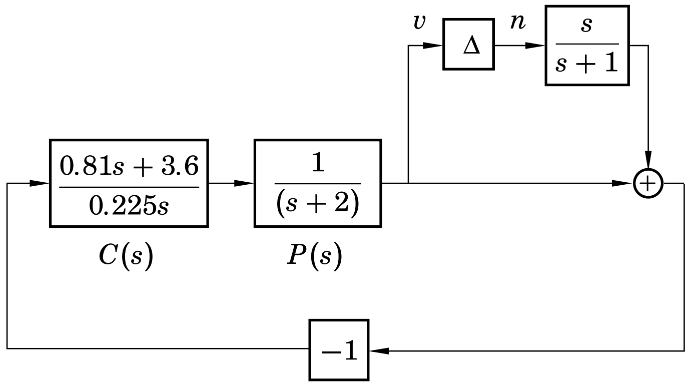



**Figure 3.4** System for Problem 3.4.

**a.** What is the largest possible bound on the gain of $\Delta$, for which the closed-loop system is stable, by the Small Gain Theorem?

**b.** If the uncertainty $\Delta$ is replaced by a real parameter $\delta$, then for what values of $\delta$ is the closed loop stable? Compare with the gain bound in (a).

**c.** What is the difference between the uncertainty model in this problem and the one in Figure 3.2?

**3.5** A continuous-time stochastic process $y(t)$ has the power spectrum $\Phi_y(\omega)$. The process can be represented by a linear filter $G(s)$ that has unit-intensity white noise $v$ as input. Determine the linear filter when

**a.**

$$\Phi_y(\omega) = \frac{a^2}{\omega^2 + a^2}, \quad a > 0$$

**b.**

$$\Phi_y(\omega) = \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}, \quad a, b > 0$$

**3.6** Consider a missile travelling in the air. It is propelled forward by a jet force $u$ along a horizontal path. The coordinate along the path is $z$. We assume that there is no gravitational force. The aerodynamic friction force is described by a simple model as

$$f = k_1 \dot{z} + v,$$

where $v$ are random variations due to wind and pressure changes. Combining this with Newton's second law, $m\ddot{z} = u - f$, where $m$ is the mass of the missile, gives the input-output relation

$$\ddot{z} + \frac{k_1}{m}\dot{z} = \frac{1}{m}(u - v).$$

**a.** Express the input-output relation in state-space form.

**b.** The disturbance $v$ has been determined to have the spectral density

$$\Phi_v(\omega) = k_0 \frac{1}{\omega^2 + a^2}, \quad k_0, a > 0$$

Expand your state-space description so that the disturbance input can be expressed as white noise.

**3.7** (*) This problem builds on Problem 3.6.

**a.** Assume that the position measurement is distorted by an additive error $n(t)$,

$$y(t) = z(t) + n(t)$$

Write down the state-space equations for the system, assuming that $n(t)$ is white noise with intensity 0.1, i.e. $\Phi_n(\omega) = 0.1$.

**b.** Solve the same problem, this time with

$$\Phi_n(\omega) = 0.1 \frac{\omega^2}{\omega^2 + b^2}, \quad b > 0$$

**c.** Solve the problem with

$$\Phi_n(\omega) = 0.1 \frac{1}{\omega^2 + b^2}, \quad b > 0$$

**3.8** **(*)** Consider an electric motor with the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

from input current to output angle.

There are two different disturbance scenarios:

(i) $Y(s) = G(s)(U(s) + W(s))$

(ii) $Y(s) = G(s)U(s) + W(s)$

In both cases, $\dot{w}(t) = v(t)$, where $v(t)$ is a unit disturbance, e.g., an impulse.

**a.** Draw a block diagram of the two cases.

**b.** Convert both cases into state-space form.

**c.** Give a physical interpretation of $w(t)$ in both cases.

# Solutions to Exercise 3. Disturbance Models and Robustness

**3.1** Let $G(s) = \dfrac{1}{s+1}$.

**a.** The system is guaranteed to be stable according to the Small Gain Theorem if $\|G\| \cdot \|\Delta\| < 1$. We see that $\|G\| = 1$ (calculate $\sup_\omega |G(i\omega)|$ or use `norm(G,inf)` or look at the Bode magnitude plot of $G$), so the system is guaranteed to be stable if $\|\Delta\| < 1$.

**b.** The transfer function of the closed-loop system is

$$\frac{G(s)}{1+G(s)K} = \frac{1}{s+1+K}$$

The pole is located in $s = -(1+K)$, so the system is stable for any gain $K > -1$. We can compare this to the result in **a**, which only guarantees that the system is stable when the gain $\|\Delta\| < 1$.

The different results arise from the fact that the Small Gain Theorem is conservative in nature, i.e., it gives a *sufficient* condition on stability, but that condition may not be *necessary*. The theorem makes no *a priori* assumptions on $\Delta$, which may be any system and not just a constant scalar as in **b**. Looking at the closed-loop poles of a known LTI system, on the other hand, shows exactly when the system is stable.

**3.2** **a.** Block diagrams of the original and the rewritten closed-loop system are shown in Figure 3.1. We have

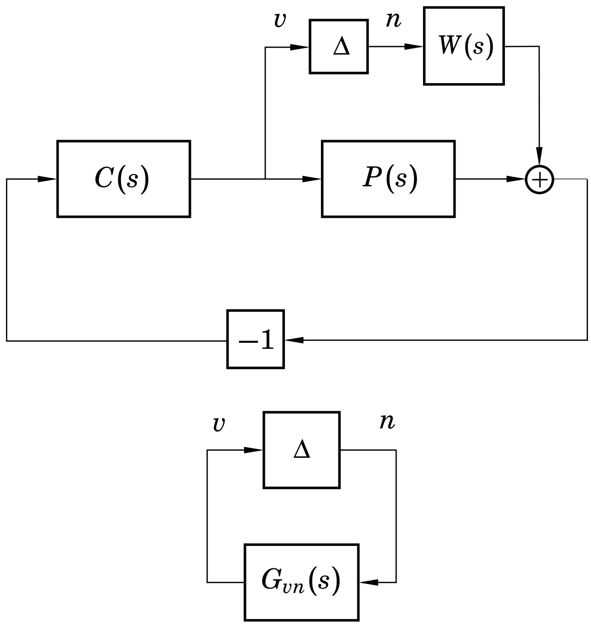



**Figure 3.1** Systems for Problem 3.2.

$$C(s) = \frac{2s+2}{s} \qquad P(s) = \frac{1}{(s+1)^2} \qquad W(s) = \frac{s}{s+2}$$

$$G_{vn}(s) = -\frac{C(s)W(s)}{1+P(s)C(s)} = -\frac{2s^4+6s^3+6s^2+2s}{s^4+4s^3+7s^2+8s+4} = -\frac{2s^3+4s^2+2s}{s^3+3s^2+4s+4}$$

Matlab commands:

```
>> s = tf('s');
>> C = 2*(s+1)/s
>> P = 1/(s+1)^2
>> W = s/(s+2);
>> Gvn = -feedback(C,P)*W;
```

**b.** The gain of $G_{vn}$ is equal to 2.63. This corresponds to the peak magnitude in the Bode diagram of Figure 3.2.

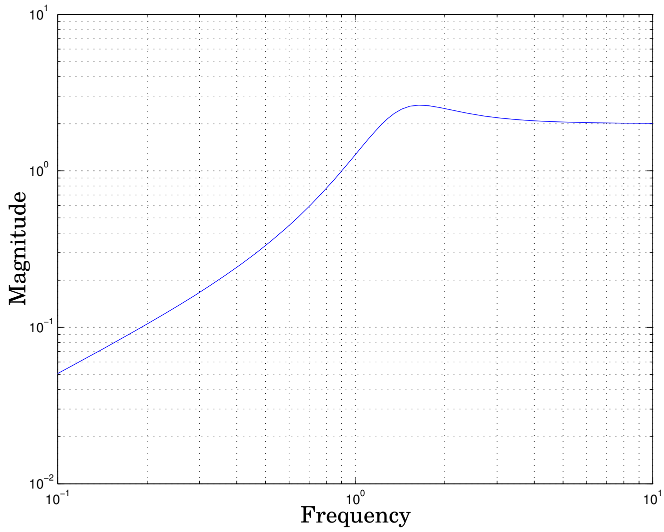



**Figure 3.2** Bode magnitude diagram for $G_{vn}(s)$.

**c.** The Small Gain Theorem shows stability for all perturbations, $\Delta$, satisfying

$$\|\Delta\| \cdot \|G_{vn}\| < 1$$

The closed-loop system is therefore stable for all perturbations $\Delta$ with

$$\|\Delta\| < 1/\|G_{vn}\| = 0.38$$

Matlab commands:

```
>> norm(Gvn, inf)
>> 1/ans
```

**d.** Process models are often more accurate in the low-frequency range. As the frequency increases there is usually excitation of higher-order dynamics and non-linearities in the real process, which are not covered by the model. The weighting factor $\frac{s}{s+2}$ in this example is used to indicate that the uncertainty is small for low frequencies (below approximately $\omega = 2$).

**3.3** **a.** Matlab commands:

```
>> s = tf('s');
>> P = 1/(s+2);
>> C = (0.81*s+3.6)/(0.225*s)
>> G = feedback(C*P,1);
>> pole(G)

ans =

  -2.8000 + 2.8566i
  -2.8000 - 2.8566i
```

**b.** The transfer functions are

$$Z = \frac{1}{1+CP}V + \frac{P}{1+CP}D - \frac{CP}{1+CP}N + \frac{CP}{1+CP}R$$

$$Z = SV + SPD + T(R - N)$$

$$Z(s) = \begin{pmatrix} S(s) & S(s)P(s) & T(s) & -T(s) \end{pmatrix} \begin{pmatrix} V(s) \\ D(s) \\ R(s) \\ N(s) \end{pmatrix}$$

Matlab commands:

```
>> T = feedback(C*P,1);
>> S = 1-T;
>> bode(T)
>> hold on
>> bode(S)
>> S

Transfer function:
   s^2 + 2 s
----------------
s^2 + 5.6 s + 16

>> T

Transfer function:
   3.6 s + 16
----------------
s^2 + 5.6 s + 16
```

**c.** In the bode plot of the sensitivity function, we see that $|S(i0.5)| = -23.8$ dB $=$ 0.0646

Matlab commands:

```
>> abs(freqresp(S,0.5))

ans =

    0.0644
```

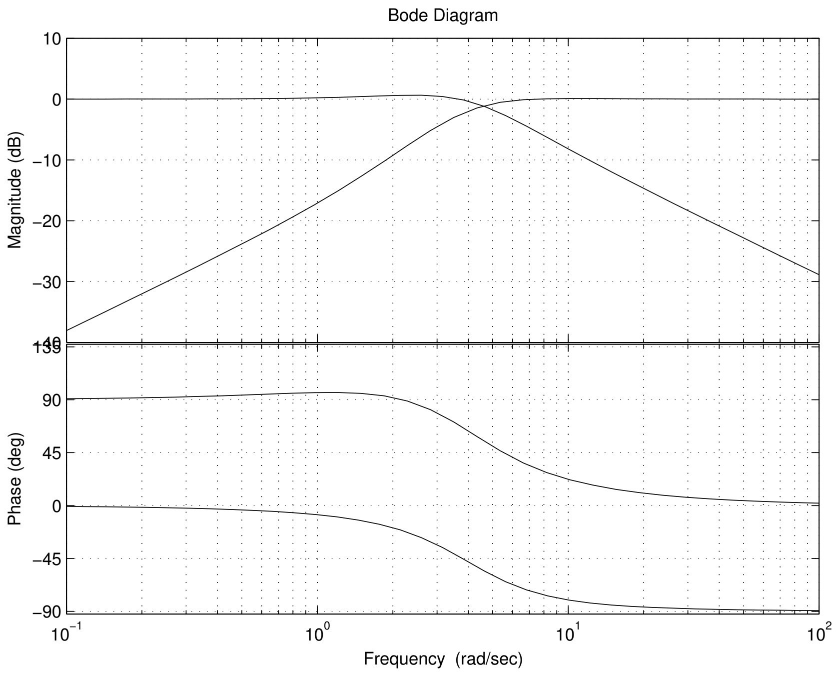



**Figure 3.3** Bode diagrams of $S$ and $T$ in Problem 3.3.

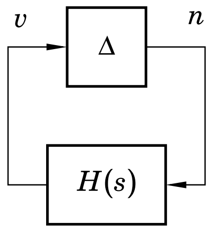



**Figure 3.4** Rewritten closed-loop system for Problem 3.4(a).

**d.** We have $\omega = 2\pi 50$ rad/s $= 314.16$ rad/s. In the bode plot of the complementary sensitivity function, we see that $|T(i314.16)| = -38.8$ dB $= 0.0115$

We have very good attenuation of both load disturbances and measurement noise.

**3.4** **a.** The transfer function from $n$ to $v$ as seen in Figure 3.4 can be written as $H = \frac{-PCW}{1+PC}$ according to the following Matlab commands:

```
>> s = tf('s');
>> W = s/(s+1);
>> P = 1/(s+2);
>> C = (0.81*s+3.6)/(0.225*s);
>> H = -feedback(P*C,1)*W;
>> norm(H,inf)
 ans =
     1.0072
>> lower_bound = 1/ans
 ans =
     0.9928
```

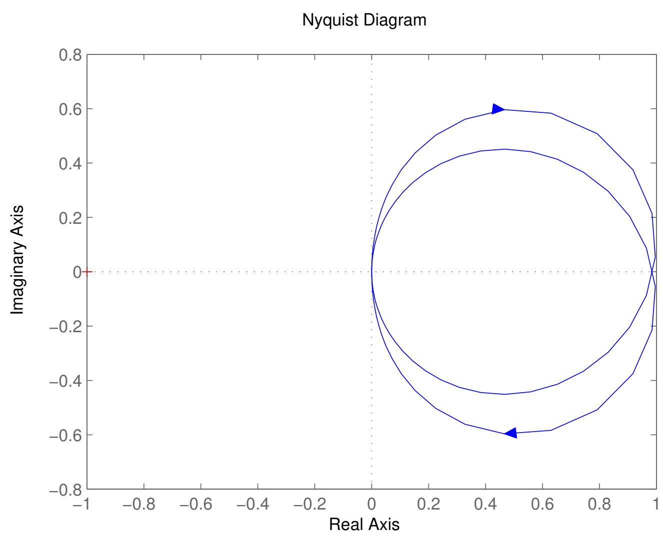



**Figure 3.5** Nyquist plot of $-H(s)$ in Problem 3.4(b).

**b.** We know that $\Delta = \delta$, a real number. Looking at Figure 3.4 we see that we can apply the Nyquist Criterion to analyze the closed-loop stability.

From the Nyquist Plot of $-H(s)$ in Figure 3.5, we see that the closed loop is stable for all $\delta \geq 0$. For negative $\delta$:s, the closed loop will become unstable once the bubble formed by the Nyquist Curve has grown so large that $-1$ is no longer on its outside. We find this value to be $-\delta = 1.0119$ from the gain margin of $-H$. Thus, the system is stable when $\delta > -1.0119$.

The Small Gain Theorem is easy to use, but it can be conservative, since there is no prior assumptions on structure of uncertainty. With more information about the uncertainty, the bounds can be less conservative and we can allow all positive values of $\delta$ as well.

Matlab code:

```
>> nyquist(-H)
>> allmargin(H)
ans =
     GainMargin: 1.0119
    GMFrequency: 2.3256
    PhaseMargin: [-4.4628 -18.9141]
    PMFrequency: [2.5211 3.1922]
    DelayMargin: [2.4614 1.8649]
    DMFrequency: [2.5211 3.1922]
         Stable: 1
```

**c.** In the referenced figure the uncertainty is *added* to the process. This is called an *additive uncertainty*, ie. $P + \Delta$. Here, the uncertainty is multiplied to the output signal, giving a *multiplicative uncertainty*, ie. $P(1 + \Delta)$. In this type of model, the uncertainty is proportional to the process gain.

**3.5** $\Phi_y(\omega)$ is an even, scalar, non-negative function. Thus we can factor it into

$$\Phi_y(\omega) = G(i\omega)\Phi_v(\omega)G(-i\omega)$$

where $G(s)$ has its poles and zeroes in the left half-plane and $\Phi_v(\omega) = 1$ (white noise).

**a.**

$$\Phi_y(\omega) = \frac{a^2}{\omega^2 + a^2}\Phi_e(\omega) = \frac{a}{i\omega + a} \cdot \frac{a}{-i\omega + a}$$

So the linear filter is

$$G(s) = \frac{a}{s+a}$$

**b.** In the same way, we get

$$\Phi_y(\omega) = \frac{a^2b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}\Phi_e(\omega)$$

$$= \frac{ab}{(i\omega + a)(i\omega + b)} \cdot \frac{ab}{(-i\omega + a)(-i\omega + b)}$$

$$\Rightarrow \quad G(s) = \frac{ab}{(s+a)(s+b)}$$

**3.6 a.** To make a state-space description, we let $x_1 = z$, $x_2 = \dot{z} \quad \Longrightarrow$

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = \frac{1}{m}(u - k_1 x_2 - v).$$

In matrix form:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{k_1}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u + \begin{pmatrix} 0 \\ -\frac{1}{m} \end{pmatrix} v,$$

$$z = \begin{pmatrix} 1 & 0 \end{pmatrix} x.$$

**b.** We want to find a filter $H$ such that

$$\Phi_v(\omega) = |H(i\omega)|^2 \Phi_e(\omega)$$

Thus $H(s) = \frac{\sqrt{k_0}}{s+a}$, which is equivalent to $\dot{v} + av = \sqrt{k_0}\, e$.
Adding a new state $x_3 = v$ to the state-space description, gives

$$\dot{x}_3 = -ax_3 + \sqrt{k_0}e$$

and

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{k_1}{m} & -\frac{1}{m} \\ 0 & 0 & -a \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ \sqrt{k_0} \end{pmatrix} e$$

$$z = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} x, \quad \Phi_e(\omega) = 1$$

**3.7** **a.** With $\{A, B, C, N\}$ according to the solution of problem 3.6, we have

$$\begin{aligned}\dot{x} &= Ax + Bu + Ne \\ y &= Cx + n\end{aligned}$$

where $n$ has spectral density $\Phi_n = 0.1$.

**b.** A noise signal with the specified spectral density is given by the output of a linear system with white noise input that has spectral density $\Phi_{w_n} = 0.1$. The transfer function of the system is

$$G_n(s) = \frac{s}{s+b} = \frac{s+b-b}{s+b} = 1 - \frac{b}{s+b}$$

In state-space form this can be expressed as

$$\begin{aligned}\dot{x}_4 &= -bx_4 + bw_n \\ n &= -x_4 + w_n\end{aligned}$$

Combining the noise model with our original system gives the expanded state-space description:

$$\begin{aligned}\dot{x} &= \begin{pmatrix} A & 0 \\ 0 & -b \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix} \\ y &= \begin{pmatrix} C & -1 \end{pmatrix} x + w_n, \quad \Phi_{\omega_n} = 0.1\end{aligned}$$

Note that the disturbance can be described using a transfer function and white noise of any spectral density. For instance, it is often convenient to assume white noise with a spectral density of 1. In this case, the transfer function of the system would be

$$G_n(s) = \frac{\sqrt{0.1}s}{s+b}$$

The expanded state space description would then need to be adjusted to account for this.

**c.** Now, the transfer function of the noise model is $G_n(s) = \frac{1}{s+b}$. In state-space form, this is

$$\dot{x}_4 + bx_4 = w_n.$$

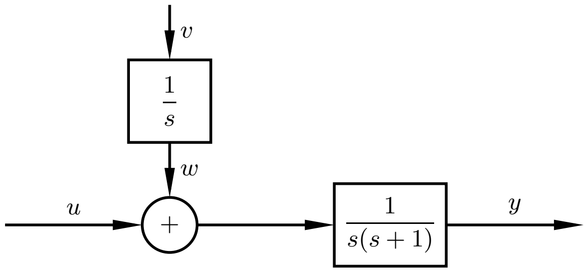
The expanded system becomes

$$\begin{aligned}\dot{x} &= \begin{pmatrix} A & 0 \\ 0 & -b \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix} \\ y &= \begin{pmatrix} C & 1 \end{pmatrix} x, \quad \Phi_{\omega_n} = 0.1\end{aligned}$$

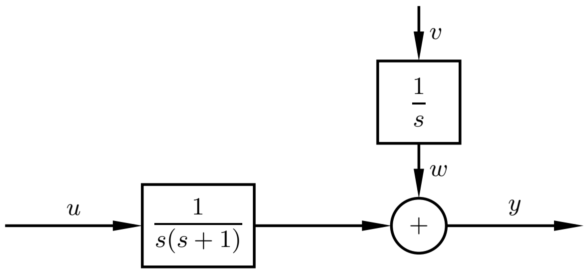
As in subproblem b, the disturbance can be described using a transfer function and white noise of any spectral density. Assuming white noise with a spectral density of 1, the transfer function of the system would be

$$G_n(s) = \frac{\sqrt{0.1}}{s+b}$$

**3.8** **a.** (i)




(ii)




$v(t)$ is a unit disturbance

**b.** (i)

$$\dot{x} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}^{A} x + \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{B} u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$
$$y = \underbrace{\begin{pmatrix} 1 & 0 & 0 \end{pmatrix}}_{C} x.$$

(ii)

$$\dot{x} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^{A} x + \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^{B} u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$
$$y = \underbrace{\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}}_{C} x.$$

**c.** (i) $w(t)$ could be an offset current on the input to the motor, and/or a step disturbance in the load.

(ii) In this case $w(t)$ is a measurement disturbance, i.e., an additive error (constant) in the angle measurement. It could also be interpreted as a load disturbance on the process output. A controller could remove the effect from a load disturbance on the process output, but not a constant measurement disturbance, so the interpretation makes a difference.