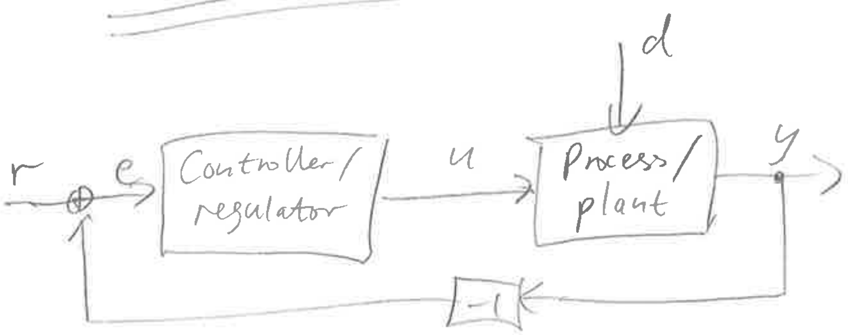
Basic control review lecture

Outline: 0. Basics
1. System representations
2. Analysis of feedback loops
3. State-space design
4. Frequency-domain design

0. Basics



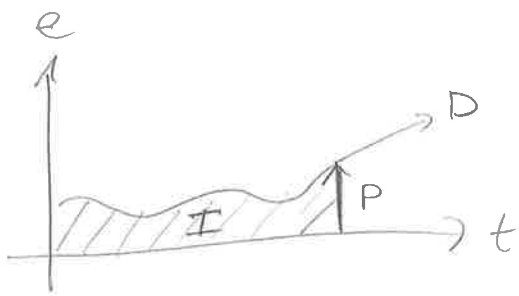
r = reference / setpoint
y = measurement signal
u = control signal
d = disturbances
e = control error

Simple controllers: P-controller: $u(t) = K e(t)$

PI-ctrl: $u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau\right)$

PID-ctrl: $u(t) = K\left(e(t) + \frac{1}{T_i}\int_0^t e(\tau)d\tau + T_d\frac{de(t)}{dt}\right)$

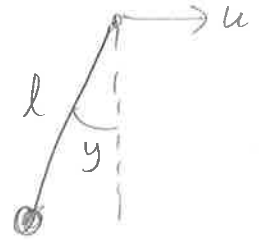
- 3 parameters to tune (K, T_i, T_d)
- I-part needed to remove stationary errors
- D-part useful for plants with inertia (ex. temp. control)



"past - present - future"

1. System representations

Ex: Pendulum



Nonlinear eq. of motion:

$$\frac{d^2y}{dt^2} + \frac{g}{l}\sin y = u\cos y$$

- Linear diff. eq: $\ddot{y} + \frac{g}{l}y = u$

- State-space model: Introduce states $x_1 = y$, $x_2 = \dot{y}$ $\Rightarrow$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}x_1 + u \\ y = x_1 \end{cases} \Leftrightarrow \qquad \begin{aligned} \dot{x} &= \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= [1 \quad 0]\, x + 0u \end{aligned}$$

- Transfer function

1) Laplace transform of linear diff eq:

$$s^2Y + \frac{g}{l}Y = U$$

$$Y = \underbrace{\frac{1}{s^2 + \frac{g}{l}}}_{=G(s)} U$$

2) Using formula $G(s) = C(sI-A)^{-1}B + D$

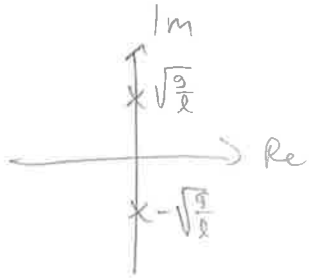
$$= \frac{1}{s^2 + \frac{g}{l}}$$

- Pole-zero map

Poles: $s^2 + \frac{g}{l} = 0$

$$s = \pm i\sqrt{\frac{g}{l}}$$

(marginally stable)

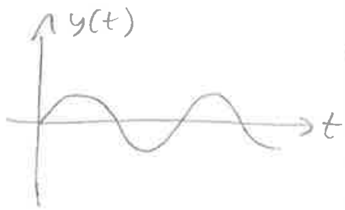


- Impulse response

$u(t) = \delta(t) \Rightarrow U(s) = 1$

$Y(s) = G(s)U(s) = G(s)$

$y(t) = \sqrt{\frac{l}{g}}\sin\left(\sqrt{\frac{g}{l}}\,t\right)$

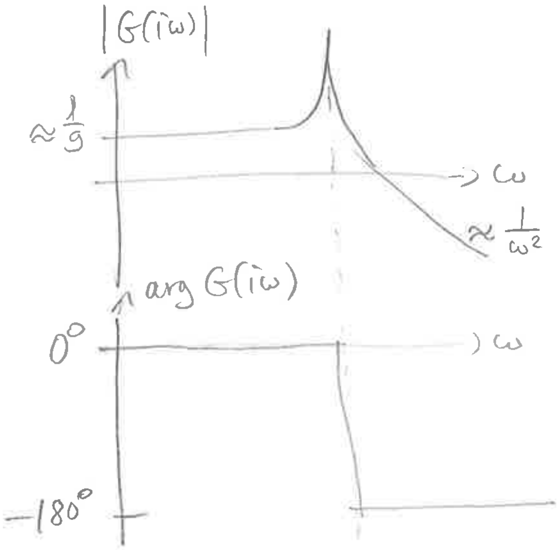


- Frequency function: $G(i\omega) = \dfrac{1}{(i\omega)^2 + \frac{g}{l}}$

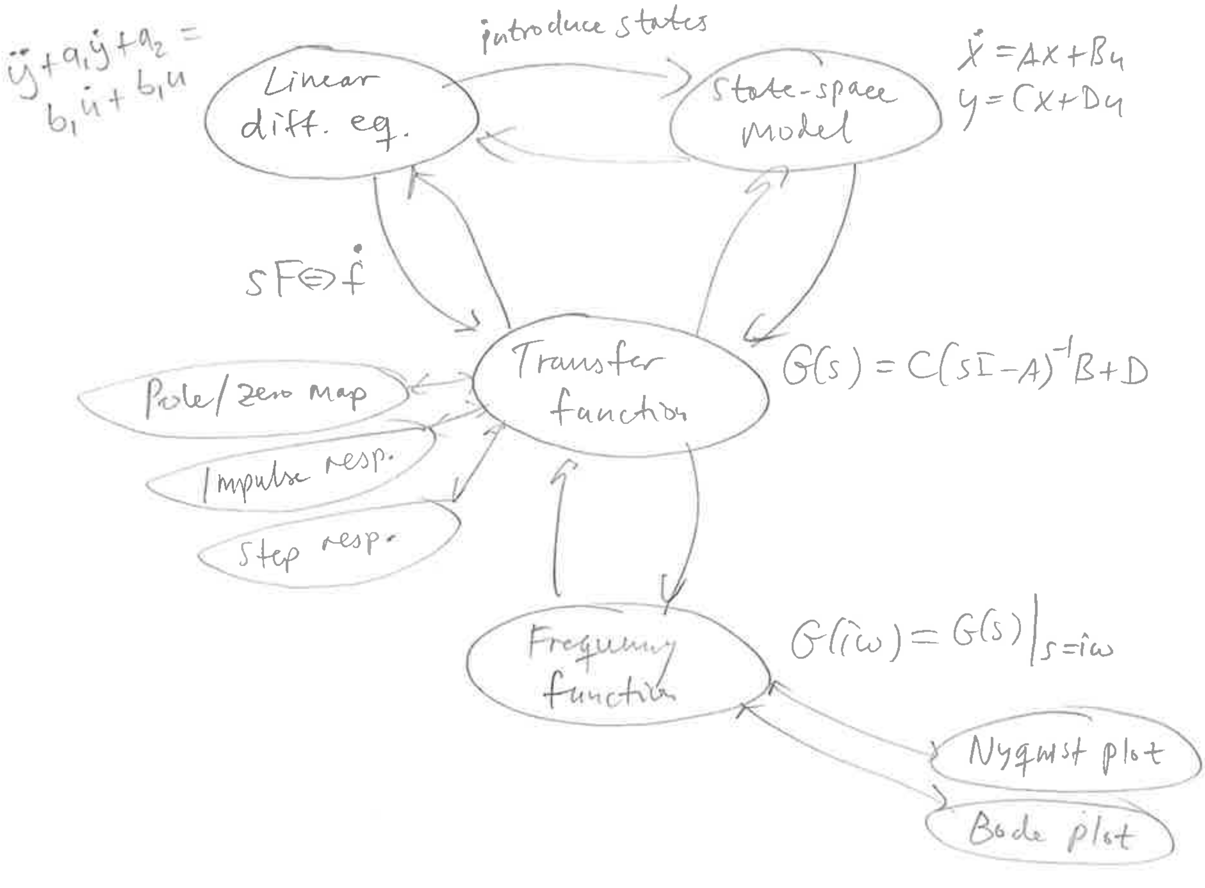
$u = \sin\omega t \Rightarrow$

$y = |G(i\omega)|\sin(\omega t + \arg G(i\omega))$

– Bode plot

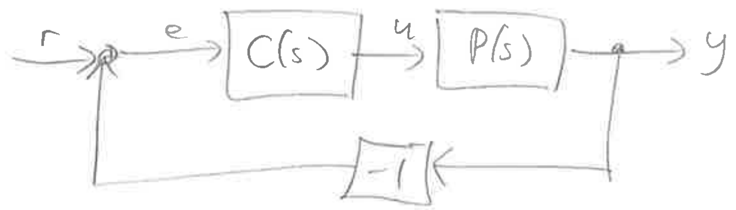


Summary of representations for linear systems



2. Analysis of feedback loops

Simple loop:



Transfer function of closed loop from r to y:

$$Y = P\cdot C\cdot(R-Y) \Rightarrow Y = \underbrace{\frac{PC}{1+PC}}_{=G_{yr}(s)} R$$

<u>Stability:</u> <u>Ex</u> Assume $P = \frac{1}{s^2+1}$ and $C = K$ (P-ctrl).

Can we stabilize the system?

$$G_{yr} = \frac{PC}{1+PC} = \frac{K}{\underbrace{s^2+1+K}_{\text{characteristic polynomial}}}$$

[Second-order polynomial $s^2+a_1 s+a_2$ has stable roots <u>if and only if</u> $a_1>0$ and $a_2>0$.

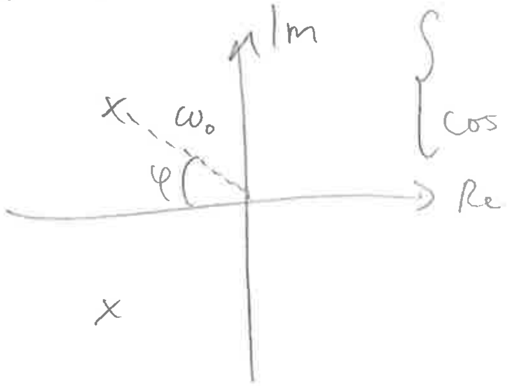
<u>Not</u> possible to obtain asymptotic stability!

Try PD-controller: $C = K(1+sT_d) \Rightarrow$

$$G_{yr} = \frac{PC}{1+PC} = \frac{K(1+sT_d)}{s^2+KT_d s+K+1}$$

As. stable $\Leftrightarrow$ $KT_d>0$ and $K+1>0$

<u>Pole placement:</u> We can arbitrarily place the poles by selecting K and T_d in the example.

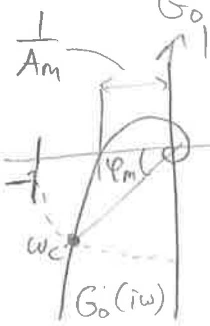


ω_0 = speed (rad/s)

$\cos\varphi$ = damping

Nyquist criterion: Plot Nyquist curve of open-loop system

$G_0(s) = P(s)C(s)$



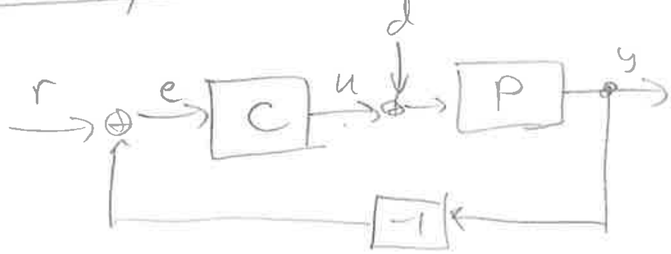
A_m = amplitude margin

φ_M = phase margin

ω_c = cross-over frequency

Then Closed loop stable if Nyquist curve does not encircle -1.

Stationary errors



Error: $E = \frac{1}{1+PC} R - \frac{P}{1+PC} D$

Assume $\begin{cases} r(t) = r_0 \\ d(t) = d_0 \end{cases} \Rightarrow \begin{cases} R(s) = \frac{r_0}{s} \\ D(s) = \frac{d_0}{s} \end{cases}$

$$\Rightarrow E = \frac{1}{1+PC}\frac{r_0}{s} - \frac{P}{1+PC}\frac{d_0}{s}$$

If closed loop stable apply final value theorem:

$$\lim_{e\to\infty} e(t) = \lim_{s\to 0} sE(s) = \frac{r_0}{1+P(0)C(0)} - \frac{P(0)d_0}{1+P(0)C(0)}$$

- To eliminate error due to r, $P(0)C(0)$ should have high (infinite) gain.
- To eliminate error due to d, $C(0)$ should have high (inf.) gain

$\Rightarrow$ Need integrator $\left(\frac{1}{s}\right)$ in $C(s)$!

Sensitivity

$$S = \frac{1}{1+PC} = \text{sensitivity fcn}$$

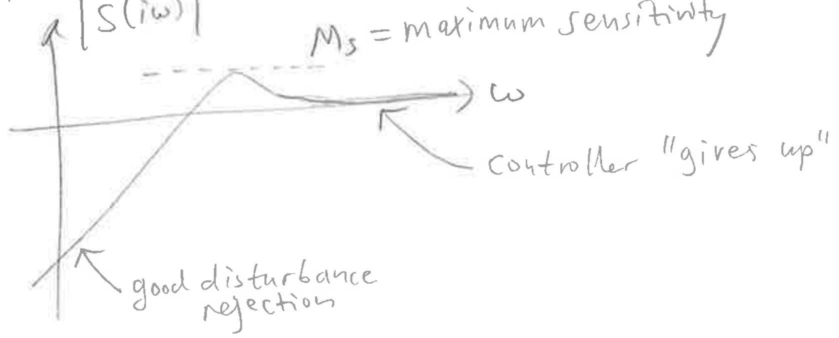
$$T = \frac{PC}{1+PC} = \text{complementary sensitivity fcn}$$

Open-loop response to disturbance : $Y_{ol} = P \cdot D$

Closed-loop — " — : $Y_{cl} = \frac{P}{1+PC} D = S\, Y_{ol}$

S describes how the feedback amplifies (or attenuates) the effect of disturbances for different frequencies.

Typical sensitivity functions



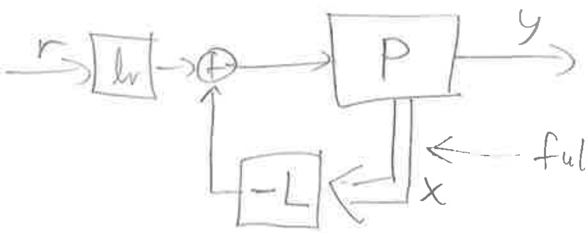
3. State-space design

General methodology to design stabilizing controller for any $P(s)$

Plant: $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$ (A $n \times n$ matrix)

Two components, designed independently:
- state feedback
- observer (Kalman filter)

State feedback



full state vector, unrealistic

Control law: $u = -Lx + l_r r$, $L = [l_1 \; l_2 \; \dots \; l_n]$

$\Rightarrow$ closed loop: $\dot{x} = (A - BL)x + B l_r r$

Eigenvalues: $\det(sI - A + BL) = 0$

- Select L to give desired eigenvalues/poles
- Choose l_r to give $G_{yr}(0) = 1$ (unit stationary gain)

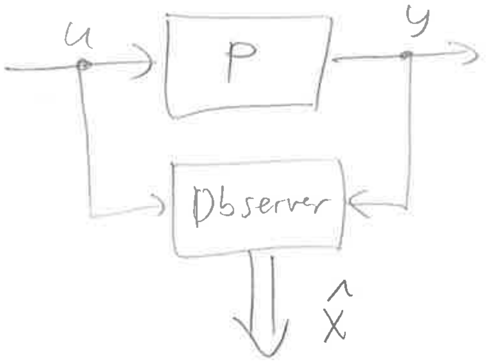
Controllability: Poles can be placed arbitrarily if system is controllable $\Leftrightarrow$

$$W_c = [B \; AB \; A^2B \; \dots \; A^{n-1}B]$$

should have full rank ($= n$)

Observer

Reconstruct process state by dynamical filters



Observer: $\dot{\hat{x}} = A\hat{x} + Bu + K(y - C\hat{x})$, $K = \begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix}$

Error dynamics: $\tilde{x} = x - \hat{x} \Rightarrow$

$$\dot{\tilde{x}} = (A - KC)\tilde{x}$$

Poles: $\det(sI - A + KC) = 0$

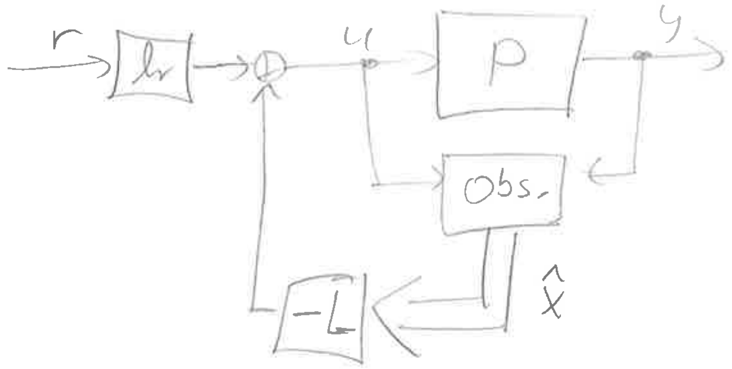
- Select K to give desired eigenvalues/poles
- Rule of thumb: Make observer poles 2 x faster than state feedback poles

Observability: Poles can be placed arbitrarily if system is observable $\Leftrightarrow$

$$W_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

should have full rank $(= n)$

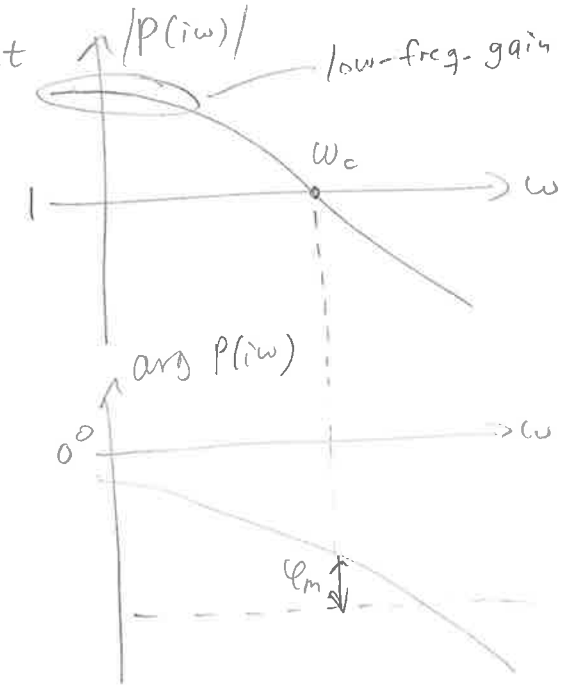
Output feedback: Connect observer and state feedback:



4. Frequency domain design

Idea: Shape the open-loop ($G_0 = CP = L$) frequency response.

Typical plant

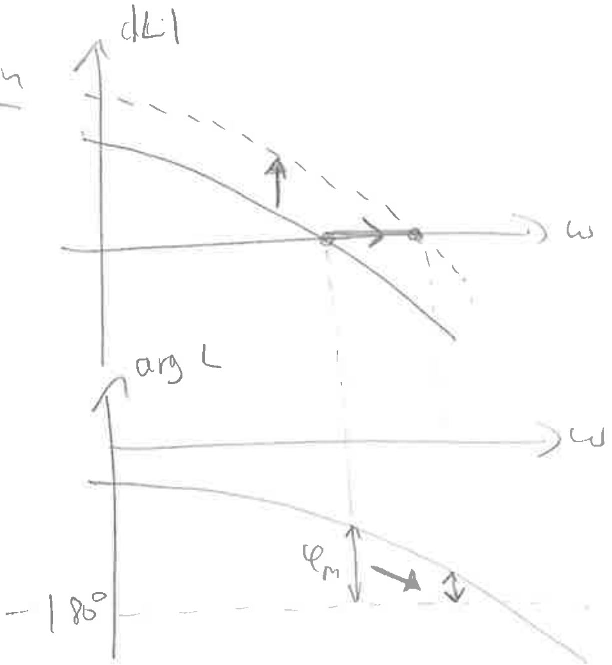


Typical goals:
- High enough low-freq. gain — disturbance rejection
- High enough crossover freq. ω_c — speed
- Large enough phase margin φ — robustness/damping

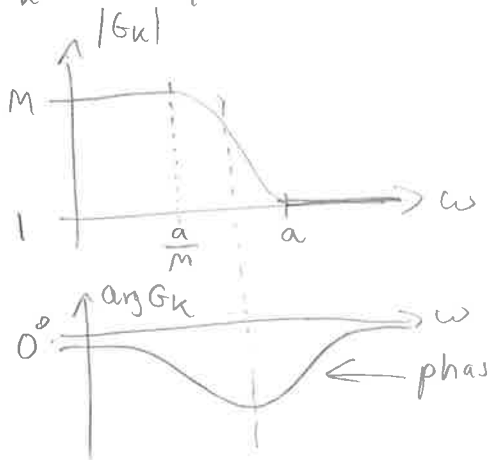
Working procedure: Construct controller $C = C_1 C_2 C_3$ by adding suitable filters/links until req. fulfilled.

- Gain K ("P controller")
  - increase speed
  - increase LF gain
  - decrease robustness
- Lag filter (fasretarderande länk)
  - increase LF gain
  - (slightly) decrease robustness
- Lead filter (fasavancerande länk)
  - increase robustness
  - affects speed + LF gain

- Increased gain



- Lag filter $G_K = M \dfrac{1+s/a}{1+sM/a}, \; M>1$

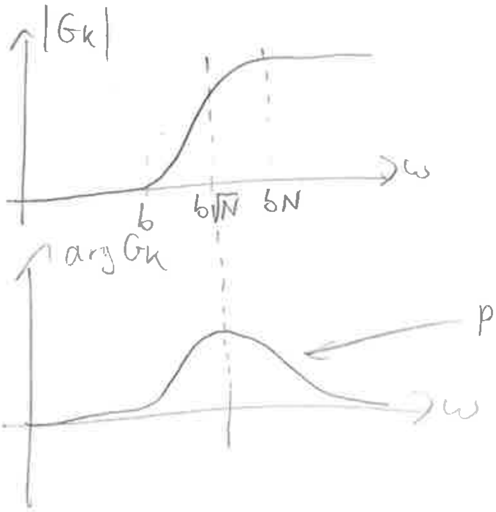


← phase lag (bad)

- LF gain increased by M
- select $a = 0.1\,\omega_c$ to avoid decreasing φ_m too much

- Lead filter $G_K = \dfrac{1+s/b}{1+s/(bN)}, \; N>1$

- Select N for desired phase lead (tables)
- Adjust b to place lead near desired ω_c



phase lead (good)