FRTN10 Multivariable Control, Lecture 15

Automatic Control LTH, 2016

Lecture 15: Course Summary

- L1-L5 Specifications, models and loop-shaping by hand
- L6-L8 Limitations on achievable performance
- L9-L11 Controller optimization: Analytic approach
- L12-L14 Controller optimization: Numerical approach

Examples

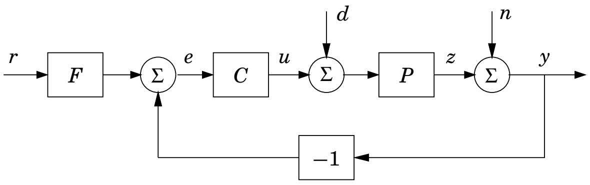
Flexible servo resonant system
Quadruple tank system multivariable (MIMO), NMP zero
Rotating crane multivariable, observer needed

DVD control resonant system, wide frequency range, (midranging)
Bicycle steering unstable pole/zero-pair
Distillation column MIMO, input-output pairing

Course Summary

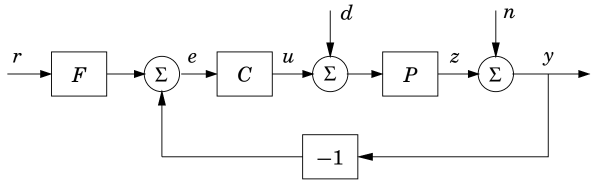
- **Specifications, models and loop-shaping**
- Limitations on achievable performance
- Controller optimization: Analytic approach
- Controller optimization: Numerical approach

2-DOF control



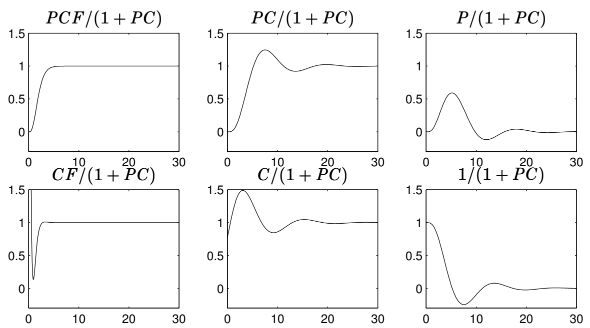
- Reduce the effects of load disturbances
- Limit the effects of measurement noise
- Reduce sensitivity to process variations
- Make output follow command signals

2DOF control



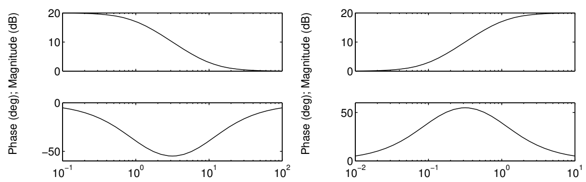
$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$
$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

Important step responses



Lag and lead filters for loop-shaping

$$C(s) = \frac{s+10}{s+1} \qquad C(s) = \frac{10(s+1)}{(s+10)}$$



MIMO systems

If C, P and F are general MIMO-systems, so called **transfer function matrices**, the **order of multiplication matters** and

$$PC \neq CP$$

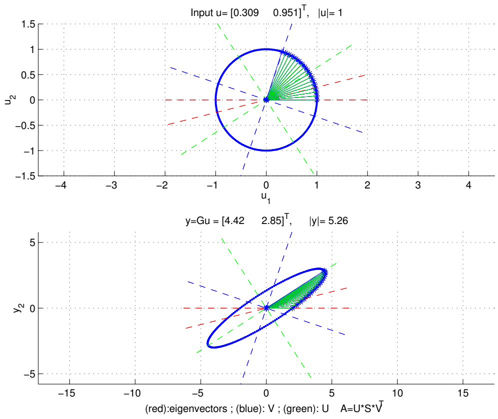
and thus we need to multiply with the inverse from the correct side as in general

$$(I+L)^{-1}M \neq M(I+L)^{-1}$$

Note, however that

$$(I+PC)^{-1}PC = P(I+CP)^{-1}C = PC(I+PC)^{-1}$$

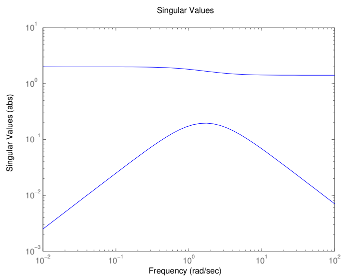
Different gains in different directions: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



Plot singular values of $G(i\omega)$ versus frequency

```
» s=tf('s')
» G=[1/(s+1) 1 ; 2/(s+2) 1]
» sigma(G) % plot singular values
```

```
% Alt. for a certain frequency:
» w = 1;
» A = [1/(i*w+1) 1; 2/(i*w+2) 1]
» [U,S,V] = svd(A)
```



Realization of multi-variable system

Example: To find state space realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+3)} \\ \frac{6}{(s+2)(s+4)} & \frac{1}{s+2} \end{bmatrix}$$

we write the transfer matrix as

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+3} \\ \frac{3}{s+2} - \frac{3}{s+4} & \frac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix}}{s+2} - \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+3} - \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \end{bmatrix}}{s+4}$$

This gives the realization

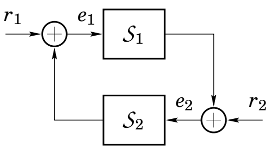
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$

The Small Gain Theorem

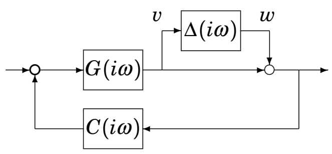
Consider a linear system $\mathcal{S}$ with input u and output $\mathcal{S}(u)$ having a (Hurwitz) stable transfer function $G(s)$. Then, the system gain

$$\|\mathcal{S}\| := \sup_u \frac{\|\mathcal{S}(u)\|}{\|u\|} \quad \text{is equal to} \quad \|G\|_\infty := \sup_\omega |G(i\omega)|$$



Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are input-output stable. If $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$, then the gain from (r_1, r_2) to (e_1, e_2) in the closed-loop system is finite.

Application to robustness analysis



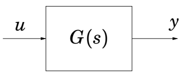
The transfer function from w to v is

$$\frac{G(i\omega)C(i\omega)}{1+G(i\omega)C(i\omega)}$$

Hence the small gain theorem guarantees closed-loop stability for all perturbations Δ with

$$\|\Delta\| < \left(\sup_\omega \left| \frac{G(i\omega)C(i\omega)}{1+G(i\omega)C(i\omega)} \right| \right)^{-1}$$

Spectral density



Assume that the stationary mean-zero stochastic process u has spectral density $\Phi_u(\omega)$. Then

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G(i\omega)^*$$

- "Any spectrum" can be generated by filtering white noise
- Finding $G(s)$ given $\Phi_y(\omega)$ is called spectral factorization

State-space system with white noise input

Given the system

$$\dot{x} = Ax + Bv, \qquad \Phi_v(\omega) = R$$

the stationary covariance of the state x is given by

$$\Pi_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega) d\omega$$

The symmetric matrix Π_x can be found by solving the Lyapunov equation

$$A\Pi_x + \Pi_x A^T + BRB^T = 0$$

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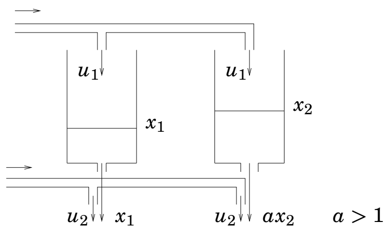
- Specifications, models and loop-shaping
- **Limitations on achievable performance**
- Controller optimization: Analytic approach
- Controller optimization: Numerical approach

Example: Two water tanks

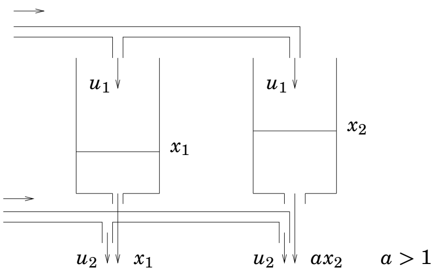
Example from Lecture 6:



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$
$$y_1 = x_1 + u_2 \qquad y_2 = ax_2 + u_2$$

Can you reach $y_1 = 1, y_2 = 2$? Can you stay there?

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

The controllability Gramian $S = \int_0^\infty \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$

is close to singular for $a \approx 1$, so it is harder to reach a desired state.

Computing the controllability Gramian

The controllability Gramian $S = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ can be computed by solving the linear system of equations

$$AS + SA^T + BB^T = 0$$

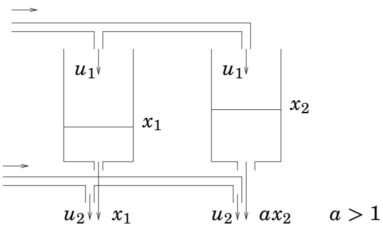
$S = S^T > 0$, i.e., S is a symmetric positive definite matrix

Example: For a 2-state system, assign

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

Multiply together and solve for s_{11}, s_{12}, s_{22} in the same way as you also do for the spectral factorization and the Riccati equations...

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix}$. Find zero from $\det G(s) = \dfrac{-s}{(s+1)(s+2)}$

There is a zero at $s = 0$! Outputs must be equal at stationarity.

Sensitivity bounds from RHP zeros and poles

Rules of thumb:

"The closed-loop bandwidth must be less than z."
"The closed-loop bandwidth must be greater than p."
"Time delays T must be less than $1/p$."

Hard bounds:

The sensitivity must be one at an unstable zero:

$$P(z) = 0 \qquad \Rightarrow \qquad S(z) := \frac{1}{1 + P(z)C(z)} = 1$$

The complimentary sensitivity must be one at an unstable pole:

$$P(p) = \infty \qquad \Rightarrow \qquad T(p) := \frac{P(p)C(p)}{1 + P(p)C(p)} = 1$$

Maximum Modulus Theorem

Assume that $G(s)$ is rational, proper and stable. Then

$$\max_{\text{Re } s \geq 0} |G(s)| = \max_{\omega \in \mathbf{R}} |G(i\omega)|$$

Corollary:

Suppose that the plant $P(s)$ has unstable zeros z_i and unstable poles p_j. Then the specifications

$$\sup_\omega |W_S(i\omega) S(i\omega)| < 1 \qquad \sup_\omega |W_T(i\omega) T(i\omega)| < 1$$

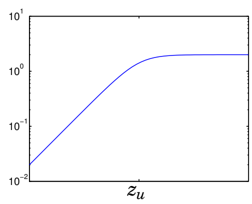
are impossible to meet with a stabilizing controller unless $\|W_S(z_i)\| < 1$ for every unstable zero z_i and $\|W_T(p_j)\| < 1$ for every unstable pole p_j.

Hard limitations from unstable zeros

If the plant has an unstable zero z_u, then the specification

$$\left| \frac{1}{1 + P(i\omega)C(i\omega)} \right| < \frac{2}{\sqrt{1 + z_u^2/\omega^2}} \qquad \text{for all } \omega$$

is impossible to satisfy.



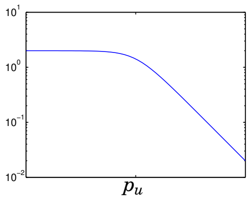
Examples: Rear-wheel steering and quadruple tank process

Hard limitations from unstable poles

If the plant has an unstable pole p_u, then the specification

$$\left|\frac{P(i\omega)C(i\omega)}{1+P(i\omega)C(i\omega)}\right| < \frac{2}{\sqrt{\omega^2/p_u^2+1}} \qquad \text{for all } \omega$$

is impossible to satisfy.



Example: Inverted pendulum

Relative Gain Array (RGA)

For a square matrix $A \in \mathbf{C}^{n\times n}$, define

$$\text{RGA}(A) := A.*(A^{-1})^T$$

where ".*" denotes element-by-element multiplication.
(For a non-square matrix, use pseudo inverse $A^\dagger$)

- The sum of all elements in a column or row is one.
- Permutations of rows or columns in A give the same permutations in RGA(A)
- RGA(A) = RGA(D_1AD_2) if D_1 and D_2 are diagonal, i.e. RGA(A) is independent of scaling
- If A is triangular, then RGA(A) is the unit matrix I.

Decoupling

Simple idea: Find a compensator so that the system appears to be without coupling ("block-diagonal transfer function matrix").

- Input decoupling $Q = PD_1$
- Output decoupling $Q = D_2P$
- "both" $Q = D_2PD_1$

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Nonmin-phase zero and unstable pole

Let $P = \widehat{P}(s-z)(s-p)^{-1}$, with $\widehat{P}$ proper and $\widehat{P}(p) \neq 0$.

Then, for stable closed loop the sensitivity function satisfies

$$\sup_\omega |S(i\omega)| \geq \left|\frac{z+p}{z-p}\right|$$

so if $p \approx z$, then the sensitivity function must have a high peak **for every controller** C.

Example: Bicycle with rear wheel steering

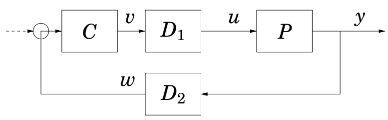
$$\frac{\theta(s)}{\delta(s)} = \frac{am\ell V_0}{bJ}\cdot\frac{(-s+V_0/a)}{(s^2-mg\ell/J)}$$

Example: RGA for a distillation column

- Find a permutation of inputs and outputs that makes RGA$(P(0))$ as close as possible to the identity matrix.
- Avoid pairings that give negative diagonal elements of RGA$(P(0))$

$$\text{RGA}(P(0)) = \begin{bmatrix} 0.2827 & -0.6111 & 1.3285 \\ 0.0134 & 1.5827 & -0.5962 \end{bmatrix}$$

To choose control signal for y_1, we apply the heuristics to the top row and choose u_3. Based on the bottom row, we choose u_2 to control y_2. Decentralized control!

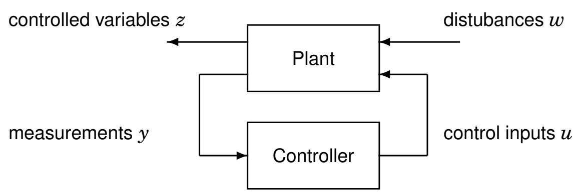


Find D_1 and D_2 so that the controller sees a "diagonal plant":

$$D_2PD_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a "decentralized" controller C with same block-diagonal structure.

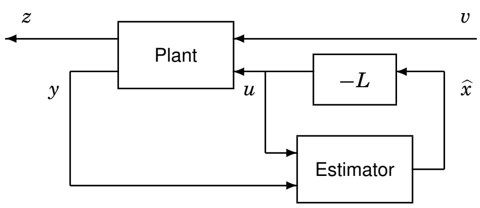
A general optimization setup



The objective is to find a controller that optimizes the transfer matrix $G_{zw}(s)$ from disturbances w to controlled outputs z.

Lecture 9-11: Problems with analytic solutions
Lectures 12-14: Problems with numeric solutions

Output feedback using state estimates



Plant: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$

Controller: $\begin{cases} \frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) + Bu(t) + K[y(t) - C\widehat{x}(t)] \\ u(t) = -L\widehat{x}(t) \end{cases}$

Linear Quadratic Gaussian (LQG) control

Given the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(k) \\ y(t) = Cx(t) + v_2(t) \end{cases} \qquad Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} > 0 \qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix} > 0$$

consider controllers of the form $u = -L\widehat{x}$ with $\frac{d}{dt}\widehat{x} = A\widehat{x} + Bu + K[y - C\widehat{x}]$. The cost function

$$\mathbf{E}\left\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right\}$$

is minimized when K and L satisfy

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \qquad L = Q_2^{-1}(SB + Q_{12})^T$$
$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T \qquad K = (PC^T + R_{12})R_2^{-1}$$

The minimal value of the cost is

$$\text{tr}(SR_1) + \text{tr}[PL^T(B^T SB + Q_2)L]$$

Tuning the weights

- A small Q_2 compared to Q_1 means that control is "cheap"
  - Resulting LQ controller will have large feedback gain
  - The plant state will be quickly regulated to zero
  - A large cost on an individual state x_i means that more effort will be spent on regulating that particular state to zero
- A small R_2 compared to R_1 means that measurements can be trusted
  - Resulting Kalman filter will have large filter gain
  - The initial estimation error will quickly converge to zero
  - A large noise covariance on an individual state x_i means that the estimation error will decay faster for that particular state

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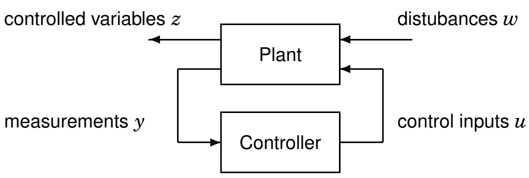
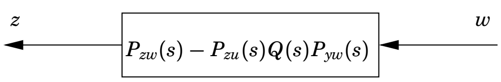
The Q-parameterization (Youla)

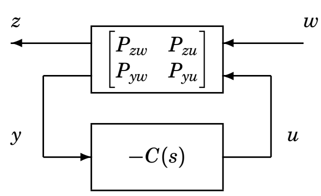


Idea for lecture 12-14:
The choice of controller generally corresponds to finding $Q(s)$, to get desirable properties of the map from w to z:



Once $Q(s)$ is determined, a corresponding controller is derived.

The Youla Parameterization



The closed-loop transfer matrix from w to z is

$$G_{zw}(s) = P_{zw}(s) - P_{zu}(s)Q(s)P_{yw}(s)$$

where

$$Q(s) = C(s)\left[I + P_{yu}(s)C(s)\right]^{-1}$$
$$C(s) = Q(s) + Q(s)P_{yu}(s)C(s)$$
$$C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1}Q(s)$$

Synthesis by convex optimization

A general control synthesis problem can be stated as a convex optimization problem in the variables $Q_0, \dots, Q_m$. The problem has a quadratic objective, with linear and quadratic constraints:

Minimize $\int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega) \overbrace{\sum_k Q_k \phi_k(i\omega)}^{Q(i\omega)} P_{yw}(i\omega)|^2 d\omega$ } quadratic objective

subject to
step response $w_i \to z_j$ is smaller than f_{ijk} at time t_k
step response $w_i \to z_j$ is bigger than g_{ijk} at time t_k } linear constraints
Bode magnitude $w_i \to z_j$ is smaller than h_{ijk} at ω_k } quadratic constraints

Once the variables $Q_0, \dots, Q_m$ have been optimized, the controller is obtained as $C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1}Q(s)$

Model reduction by balanced truncation

Consider a balanced realization of a stable system,

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + Du$$

with the lower part of the Gramian being $\Sigma_2 = \begin{bmatrix} \sigma_{r+1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$.

Replacing the second state equation by $\dot{\xi}_2 = 0$ gives the relation $0 = A_{21}\xi_1 + A_{22}\xi_2 + B_2 u$. The reduced system

$$\begin{cases} \dot{\xi}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\xi_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_r = (C_1 - C_2A_{22}^{-1}A_{21})\xi_1 + (D - C_2A_{22}^{-1}B_2)u \end{cases}$$

satisfies the error bound

$$\frac{\|y - y_r\|_2}{\|u\|_2} \le 2\sigma_{r+1} + \dots + 2\sigma_n$$