FRTN10 Multivariable Control, Lecture 14

Automatic Control LTH, 2016

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

12. Youla parameterization, internal model control
13. Synthesis by convex optimization
14. **Controller simplification**

Lecture 14 – Outline

1. Model reduction by balanced truncation
2. Application to controller simplification
3. Frequency weighted balanced truncation

Reading note: [Glad/Ljung, section 3.6]

Lecture 14 – Outline

1. Model reduction by balanced truncation
2. Application to controller simplification
3. Frequency-weighted balanced truncation

Model reduction

Mathematical modeling can lead to dynamical models of very high order

Controller synthesis using the Q-parameteritzation can lead to very high order controllers

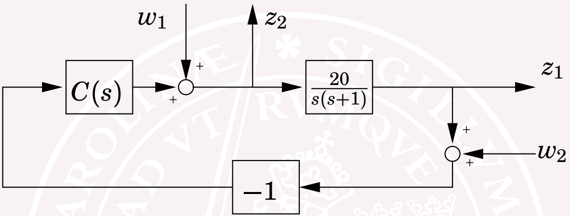
Need for systematic way to reduce the model order

In general terms we would like to achieve

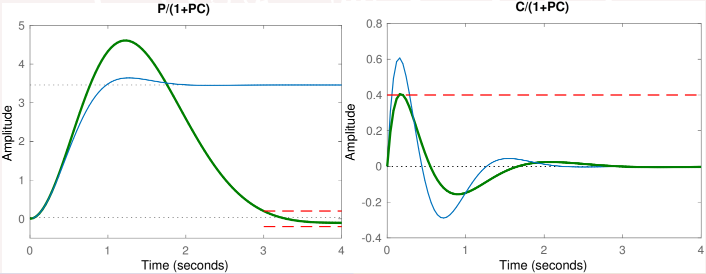
$$G_r(s) \approx G(s)$$

where $G_r(s)$ has (much) lower order than $G(s)$

Example – DC-motor



In Lecture 13 we minimized $\int_{-\infty}^{\infty} |G_{zw}(i\omega)|^2 d\omega$ subject to step response bounds on $G_{z_1 w_1}$ and $G_{z_2 w_2}$:



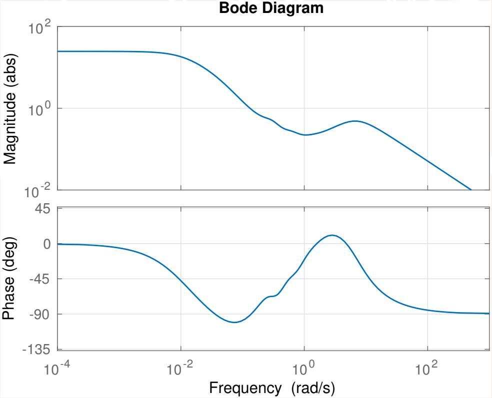
Example – DC-motor

Recall that

$$C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1}Q(s), \text{ with } Q(s) = \sum_{k=0}^{N} Q_k\phi_k(s).$$

Controller order grows with the number of basis functions ϕ_k.

Optimized controller for DC-servo has order 14. Is that really needed?



Controllability and observability

The controllability Gramian $S = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ can be computed by solving the Lyapunov equation

$$AS + SA^T + BB^T = 0$$

The observability Gramian $O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$ can be computed by solving the Lyapunov equation

$$A^T O + OA + C^T C = 0$$

We want to remove states that are both poorly controllable and poorly observable.

Gramians, looking back

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Impulse response from zero intial condition: $u_i(t) = \delta(t)$, $x(0) = 0$

$$x_i(t) = e^{At}B_i$$

$$X(t) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = e^{At}B$$

$$S_x \hat{=} \int_0^\infty X(t)X^T(t)\,dt = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt$$

Output from $u \equiv 0$ (only initial state $x(0) = x_0$)

$$y(t) = Cx(t) = Ce^{At}x_0$$

$$\int_0^\infty y(t)^Ty(t)dt = \int_0^\infty x_0^Te^{A^Tt}C^TCAtx_0dt \quad \hat{=} \quad x_0^TO_xx_0$$

Gramians, looking back

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
\qquad
x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau
$$

Impulse response from zero intial condition: $u_i(t) = \delta(t)$, $x(0) = 0$

$$
\begin{aligned}
x_i(t) &= e^{At}B_i \\
X(t) &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = e^{At}B \\
S_x \widehat{=} \int_0^\infty X(t)X^T(t)\,dt &= \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt
\end{aligned}
$$

Output from $u \equiv 0$ (only initial state $x(0) = x_0$)

$$y(t) = Cx(t) = Ce^{At}x_0$$

$$\int_0^\infty y(t)^Ty(t)dt = \int_0^\infty x_0^Te^{A^Tt}C^TCAtx_0dt \;\; \widehat{=} \;\; x_0^TO_xx_0$$

Gramians, looking back

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Impulse response from zero intial condition: $u_i(t) = \delta(t)$, $x(0) = 0$

$$x_i(t) = e^{At}B_i$$
$$X(t) = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = e^{At}B$$
$$S_x \mathrel{\widehat{=}} \int_0^\infty X(t)X^T(t)\,dt = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt$$

Output from $u \equiv 0$ (only initial state $x(0) = x_0$)

$$y(t) = Cx(t) = Ce^{At}x_0$$

$$\int_0^\infty y(t)^T y(t)dt = \int_0^\infty x_0^T e^{A^Tt}C^TCAtx_0dt \quad \widehat{=} \quad x_0^T O_x x_0$$

Balanced realizations

For a stable system (A, B, C) with Gramians S_x and O_x, the variable transformation $\xi = Tx$ gives the new state-space matrices $\widehat{A} = TAT^{-1}$, $\widehat{B} = TB$, $\widehat{C} = CT^{-1}$ and the new Gramians

$$S_\xi = \int_0^\infty e^{\widehat{A}t}\widehat{B}\widehat{B}^T e^{\widehat{A}^T t}dt = \int_0^\infty Te^{At}BB^T e^{A^T t}T^T dt = TS_xT^T$$

$$O_\xi = \int_0^\infty e^{\widehat{A}^T t}\widehat{C}^T\widehat{C}e^{\widehat{A}t}dt = \int_0^\infty T^{-T}e^{At}C^T Ce^{A^T t}T^{-1}dt = T^{-T}O_xT^{-1}$$

A particular choice of T gives $S_\xi = O_\xi = \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}}_{\Sigma}$

The corresponding realization

$$\begin{cases} \dot{\xi} = \widehat{A}\xi + \widehat{B}u \\ y = \widehat{C}x \end{cases}$$

is called a **balanced realization**.

Balanced realizations

For a stable system (A, B, C) with Gramians S_x and O_x, the variable transformation $\xi = Tx$ gives the new state-space matrices $\widehat{A} = TAT^{-1}$, $\widehat{B} = TB$, $\widehat{C} = CT^{-1}$ and the new Gramians

$$S_\xi = \int_0^\infty e^{\widehat{A}t}\widehat{B}\widehat{B}^T e^{\widehat{A}^T t}dt = \int_0^\infty Te^{At}BB^Te^{A^Tt}T^T dt = TS_xT^T$$

$$O_\xi = \int_0^\infty e^{\widehat{A}^T t}\widehat{C}^T\widehat{C}e^{\widehat{A}t}dt = \int_0^\infty T^{-T}e^{At}C^TCe^{A^Tt}T^{-1}dt = T^{-T}O_xT^{-1}$$

A particular choice of T gives $S_\xi = O_\xi = \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}}_{\Sigma}$

The corresponding realization

$$\begin{cases} \dot{\xi} = \widehat{A}\xi + \widehat{B}u \\ y = \widehat{C}x \end{cases}$$

is called a **balanced realization**.

Hankel singular values

Notice that

$$\begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} = \underbrace{(TS_xT^T)}_{\Sigma}\underbrace{(T^{-T}O_xT^{-1})}_{\Sigma} = TS_xO_xT^{-1}$$

so the diagonal elements are the eigenvalues of S_xO_x, independently of coordinate system. The numbers $\sigma_1, \ldots, \sigma_n$ are called the **Hankel singular values** of the system.

A small Hankel singular value corresponds to a state that is both weakly controllable and weakly observable. Hence, it can be truncated without much effect on the input-output behavior.

Model reduction by balanced truncation

Consider a balanced realization

$$\begin{bmatrix} \dot\xi_1 \\ \dot\xi_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + Du$$

with the lower part of the Gramian being $\Sigma_2 = \begin{bmatrix} \sigma_{r+1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$.

Replacing the second state equation by $\dot\xi_2 = 0$ gives the relation $0 = A_{21}\xi_1 + A_{22}\xi_2 + B_2 u$. The reduced system

$$\begin{cases} \dot\xi_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\xi_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_r = (C_1 - C_2A_{22}^{-1}A_{21})\xi_1 + (D - C_2A_{22}^{-1}B_2)u \end{cases}$$

satisfies the error bound

$$\frac{\|y - y_r\|_2}{\|u\|_2} \le 2\sigma_{r+1} + \cdots + 2\sigma_n$$

Example

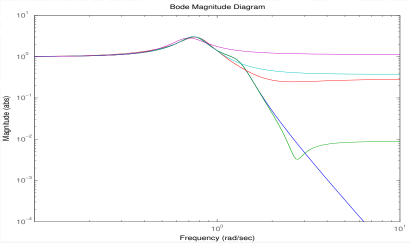
Original system: $$\frac{1-s}{s^6+3s^5+5s^4+7s^3+5s^2+3s+1}$$

Hankel singular values:

```
Sigma = [1.9837   1.9184   0.7512   0.3292   0.1478   0.0045]
```

Reduced system:

```
0.3717 s^3 - 0.9682 s^2 + 1.14 s - 0.5185
------------------------------------------
   s^3 + 1.136 s^2 + 0.825 s + 0.5185
```



Example — Heat exchanger

$$V_C \frac{dT_C}{dt} = f_C(T_{C_i} - T_C) + \beta(T_H - T_C) \qquad \text{(cold side)}$$

$$V_H \frac{dT_H}{dt} = f_H(T_{H_i} - T_H) - \beta(T_H - T_C) \qquad \text{(hot side)}$$

$u_1 = T_{C_i}$ is the in-flow temperature on the cold side
$x_1 = T_C$ is the out-flow temperature on the cold side
$u_2 = T_{H_i}$ is the in-flow temperature on the hot side
$x_2 = T_H$ is the out-flow temperature on the hot side

Numerical values:

$$\dot{x} = \begin{bmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{bmatrix} x + \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} u$$

$$y = x$$

Example — Heat exchanger

A state transformation $\xi_1 = -7.07(x_1 + x_2)$, $\xi_2 = 7.07(x_1 - x_2)$ gives the balanced realization

$$\dot{\xi} = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.41 \end{bmatrix} \xi + 0.0707 \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} u$$

$$y = 0.0707 \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \xi$$

the common controllability/observability Gramian

$$S_\xi = O_\xi = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.0122 \end{bmatrix}$$

and the reduced model

$$\dot{\xi}_1 = -0.01\xi_1 - 0.0707 \begin{bmatrix} 1 & 1 \end{bmatrix} u$$

$$y = -0.0707 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_1 + 0.0122 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u$$

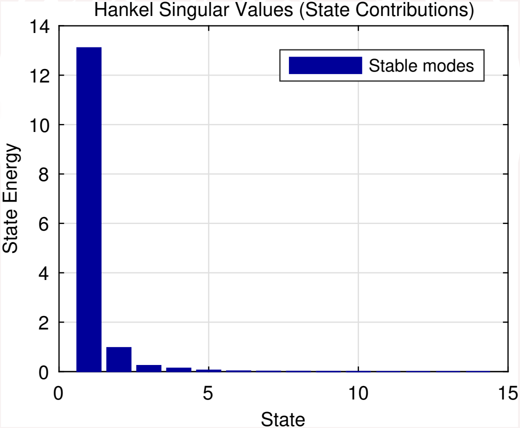
Lecture 14 – Outline

1. Model reduction by balanced truncation
2. Application to controller simplification
3. Frequency-weighted balanced truncation

Example – DC-servo

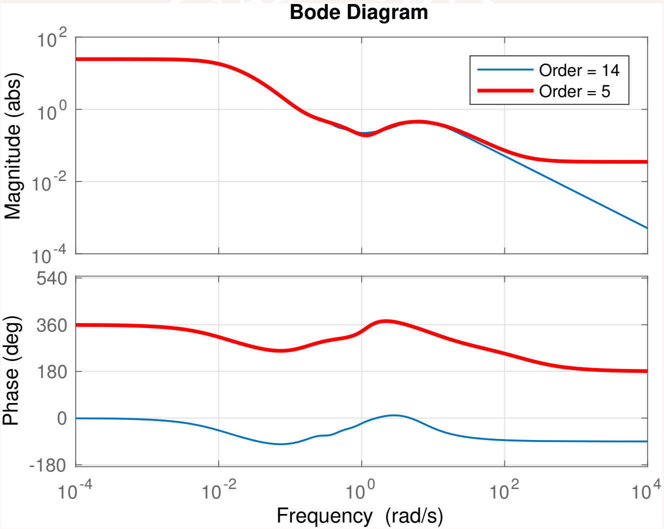
Computing the 14 Hankel singular values gives

$$\begin{bmatrix} 13.11 & 0.97 & 0.24 & 0.14 & 0.05 & 0.02 & 0.01 & \dots \end{bmatrix}$$



Example – DC-servo

Reduced controller with 5 states:



Handling unstable systems

Before model reduction, decompose the system into its stable and nonstable parts:

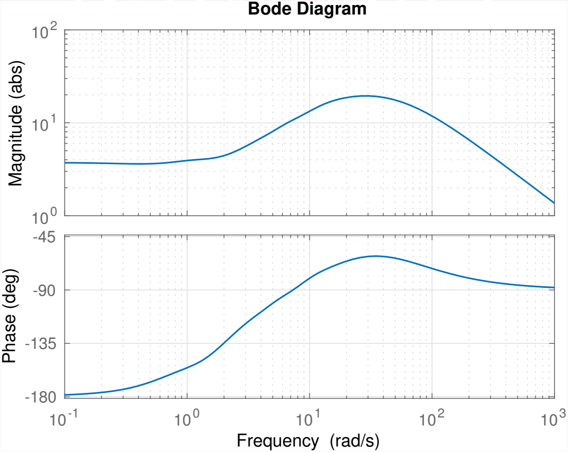
$$G(s) = G_s(s) + G_{ns}(s)$$

Perform the reduction only on $G_s(s)$; then add $G_{ns}(s)$ again

(Performed automatically by Matlab's `balreal` and `balred`)

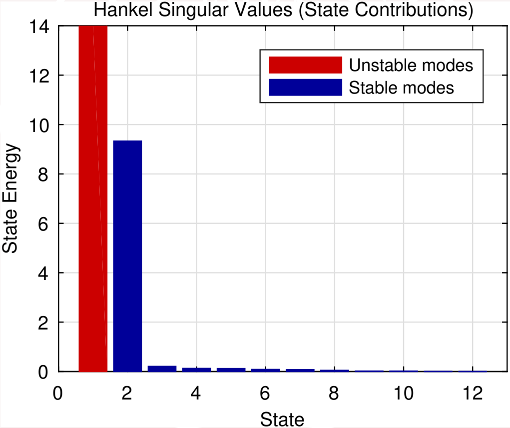
Example – Doyle–Stein (1979)

In Lecture 13 we found the following 12th order controller for Doyle–Stein's example using optimization:



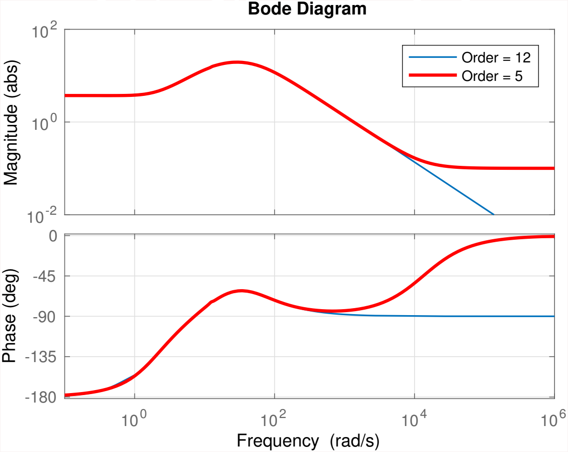
Example – Doyle–Stein (1979)

The controller has one unstable pole in 16.1. Hankel singular values:



Example – Doyle–Stein (1979)

Reduced controller with 5 states:



Lecture 14 – Outline

1. Model reduction by balanced truncation
2. Application to controller simplification
3. Frequency-weighted balanced truncation

Are all frequencies equally important?

The error bound

$$\max_{\omega} |G(i\omega) - G_r(i\omega)| = \sup_{u} \frac{\|y - y_r\|_2}{\|u\|_2} \leq 2\sigma_{r+1} + \cdots + 2\sigma_n$$

emphasizes all frequencies equally, but comparing a controller $C(s)$ with a reduced controller $C_r(s)$ in closed loop operation gives

$$|P(I + CP)^{-1}C - P(I + C_rP)^{-1}C_r| \approx |P(I + CP)^{-1}(C - C_r)|$$

Hence it is interesting to minimize the frequency weighted error

$$\max_{\omega} \left| W(i\omega)[C(i\omega) - C_r(i\omega)] \right|$$

where $W(i\omega) = P(i\omega)(I + C(i\omega)P(i\omega))^{-1}$.

Frequency-weighted balanced truncation

For model reduction with the aim to minimize

$$\max_{\omega} \left\| W_o(i\omega)[G(i\omega) - G_r(i\omega)]W_i(i\omega) \right\|$$

where

$$W_i(s) = C_i(sI - A_i)^{-1}B_i + D_i \quad G(s) = C(sI - A)^{-1}B + D \quad W_o(s) = C_o(sI - A_o)^{-1}B_o + D_o$$

find extended Gramians by solving

$$\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} S & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} + \begin{bmatrix} S & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}^T + \begin{bmatrix} BD_i \\ B_i \end{bmatrix} \begin{bmatrix} BD_i \\ B_i \end{bmatrix}^T = 0$$

$$\begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix}^T \begin{bmatrix} O & O_{12} \\ O_{12}^T & O_{22} \end{bmatrix} + \begin{bmatrix} O & O_{12} \\ O_{12}^T & O_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix} + \begin{bmatrix} C^T D_o^T \\ D_o^T \end{bmatrix} \begin{bmatrix} D_oC & D_o \end{bmatrix} = 0$$

then change coordinates to make S and O equal and diagonal before truncating the realization of $G(s)$ to get $G_r(s)$ as before.

Summary

- Low order controllers could be desirable to meet constraints on speed and memory.
- Balanced realizations can reveal less important states
- Good theoretical error bounds
- Frequency weighting essential for closed loop performance
- Reduction of unstable controllers not treated here