# FRTN10 Multivariable Control, Lecture 13

**Automatic Control LTH, 2016**

## Course Outline

L1-L5 Specifications, models and loop-shaping by hand
L6-L8 Limitations on achievable performance
L9-L11 Controller optimization: Analytic approach
L12-L14 Controller optimization: Numerical approach

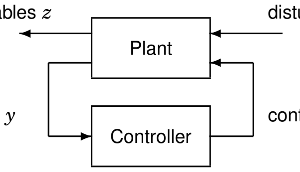
12. Youla parameterization, internal model control
13. **Synthesis by convex optimization**
14. Controller simplification

## Lecture 13 – Outline

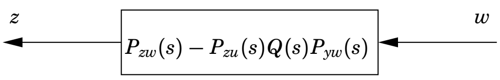
1. Examples
2. Introduction to convex optimization
3. Controller optimization using Youla parameterization
4. Examples revisited

Parts of this lecture is based on material from Boyd, Vandenberghe and coauthors. See also lecture notes and links on course homepage.

## General idea for Lectures 12–14




The choice of controller corresponds to designing a transfer matrix $Q(s)$, to get desirable properties of the following map from $w$ to $z$:




Once $Q(s)$ has been designed, the corresponding controller can be found.

## Lecture 13 – Outline

Examples

Introduction to convex optimization

Controller optimization using Youla parameterization

Examples revisited

## Example 1 (Doyle–Stein, 1979)

Given the process

$$\dot{x} = \begin{pmatrix} -4 & -3 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} -61 \\ 35 \end{pmatrix} v_1$$
$$y = \begin{pmatrix} 1 & 2 \end{pmatrix} x + v_2$$

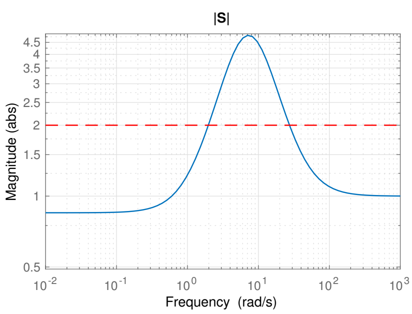
where $v_1$ and $v_2$ are independent unit-intensity white noise processes, find a controller that minimizes

$$\mathbf{E}\left\{ 80\, x^T \begin{pmatrix} 1 & \sqrt{35} \\ \sqrt{35} & 35 \end{pmatrix} x + u^2 \right\}$$

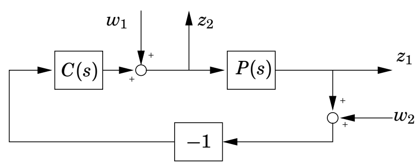
*while satisfying the robustness constraint* $M_s \le 2$

## Example 1 (Doyle–Stein, 1979)

LQG design gives a controller that does not satisfy the constraint on $S$ (see Lecture 11):




## Example 2 – DC-motor




Assume we want to optimize the closed-loop transfer matrix from $(w_1, w_2)$ to $(z_1, z_2)$,

$$G_{zw}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \end{bmatrix}$$

when $P(s) = \dfrac{20}{s(s+1)}$.

## Example 2 – DC-motor

Minimizing

$$\int_{-\infty}^{\infty} |G_{zw}(i\omega)|^2 \, d\omega$$

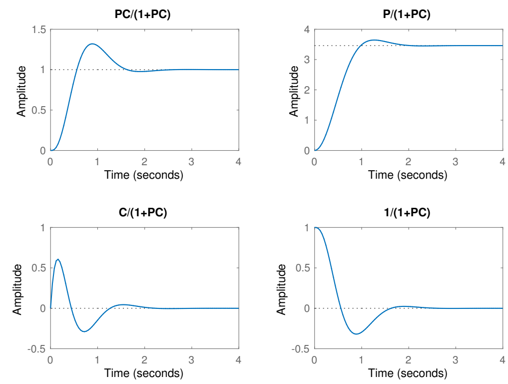
is equivalent to solving the LQG problem with (see Lecture 11)

$$A = \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \; B = N = \begin{pmatrix} 20 \\ 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 & 1 \end{pmatrix}$$

$$Q_1 = C^T C, \; Q_2 = R_1 = R_2 = 1$$

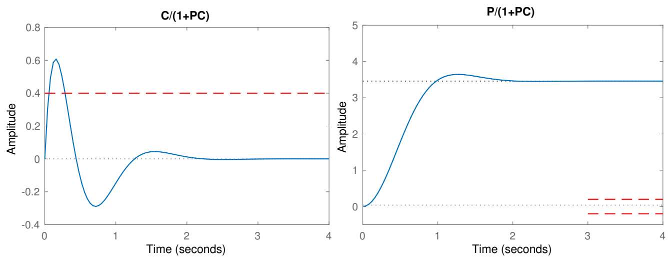
## Example 2 – DC-motor

Step responses of gang of four:




## Example 2 – DC-motor

Suppose we want to add some time-domain constraints:




- Control signal $|u| \leq 0.4$ for unit output disturbance (or setpoint change)
- Output signal $|y| \leq 0.2$ for $t \geq 3$ for unit load disturbance

## Lecture 13 – Outline

Examples

Introduction to convex optimization

Controller optimization using Youla parameterization

Examples revisited

### Least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

**solving least-squares problems**

- analytical solution: $x^\star = (A^T A)^{-1} A^T b$
- reliable and efficient algorithms and software
- computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k \times n}$); less if structured
- a mature technology

**using least-squares**

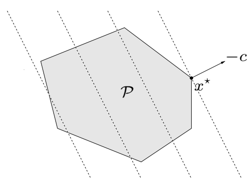
- least-squares problems are easy to recognize
- a few standard techniques increase flexibility (*e.g.*, including weights, adding regularization terms)



### Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron






### Linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

**solving linear programs**

- no analytical formula for solution
- reliable and efficient algorithms and software
- computation time proportional to $n^2 m$ if $m \geq n$; less with structure
- a mature technology

**using linear programming**

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs (*e.g.*, problems involving $\ell_1$- or $\ell_\infty$-norms, piecewise-linear functions)



### Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq b_i, \quad i = 1, \ldots, m \end{array}$$

- objective and constraint functions are convex:

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y)$$

  if $\alpha + \beta = 1, \; \alpha \geq 0, \; \beta \geq 0$

- includes least-squares problems and linear programs as special cases

**solving convex optimization problems**

- no analytical solution
- reliable and efficient algorithms
- computation time (roughly) proportional to $\max\{n^3, n^2m, F\}$, where $F$ is cost of evaluating $f_i$'s and their first and second derivatives
- almost a technology

**using convex optimization**

- often difficult to recognize
- many tricks for transforming problems into convex form
- surprisingly many problems can be solved via convex optimization



## Brief history of convex optimization

**theory (convex analysis)**: ca1900–1970

**algorithms**

- 1947: simplex algorithm for linear programming (Dantzig)
- 1960s: early interior-point methods (Fiacco & McCormick, Dikin, . . . )
- 1970s: ellipsoid method and other subgradient methods
- 1980s: polynomial-time interior-point methods for linear programming (Karmarkar 1984)
- late 1980s–now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski 1994)

**applications**

- before 1990: mostly in operations research; few in engineering
- since 1990: many new applications in engineering (control, signal processing, communications, circuit design, . . . ); new problem classes (semidefinite and second-order cone programming, robust optimization)



## Examples on R

convex:

- affine: $ax + b$ on **R**, for any $a, b \in$ **R**
- exponential: $e^{ax}$, for any $a \in$ **R**
- powers: $x^\alpha$ on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on **R**, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

concave:

- affine: $ax + b$ on **R**, for any $a, b \in$ **R**
- powers: $x^\alpha$ on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$



## Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$

affine functions are convex and concave; all norms are convex

**examples on $\mathbf{R}^n$**

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

**examples on $\mathbf{R}^{m\times n}$** ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$



## Convex optimization problem

**standard form convex optimization problem**

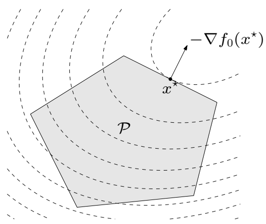
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array}$$

- $f_0$, $f_1$, . . . , $f_m$ are convex; equality constraints are affine
- problem is *quasiconvex* if $f_0$ is quasiconvex (and $f_1$, . . . , $f_m$ convex)

## Quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^T P x + q^T x + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}^n_+$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron






## Second-order cone programming

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \leq c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

($A_i \in \mathbf{R}^{n_i \times n}$, $F \in \mathbf{R}^{p\times n}$)

## Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

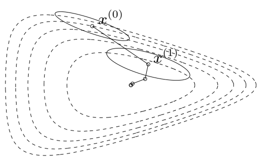
with $F_i$, $G \in \mathbf{S}^k$

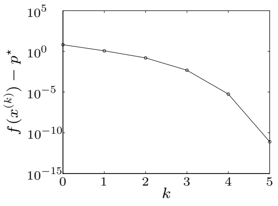
- inequality constraint is called linear matrix inequality (LMI)

## Newton's method

**given** a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.
**repeat**

1. *Compute the Newton step and decrement.*
   $\Delta x_{\mathrm{nt}} := -\nabla^2 f(x)^{-1} \nabla f(x)$; $\quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1} \nabla f(x)$.
2. *Stopping criterion.* **quit** if $\lambda^2/2 \le \epsilon$.
3. *Line search.* Choose step size $t$ by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\mathrm{nt}}$.



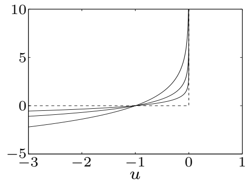

## Barrier method for constrained minimization

$$
\begin{array}{ll}
\text{minimize} & f_0(x) \\
\text{subject to} & f_i(x) \le 0 \quad 1 = 1, \dots, m \\
 & Ax = b
\end{array}
$$

**approximation via logarithmic barrier**

$$
\begin{array}{ll}
\text{minimize} & f_0(x) - (1/t) \sum_{i=1}^m \log(-f_i(x)) \\
\text{subject to} & Ax = b
\end{array}
$$

- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of $I_-$
- approximation improves as $t \to \infty$






## Lecture 13 – Outline

Examples

Introduction to convex optimization

Controller optimization using Youla parameterization

Examples revisited

## Scheme for numerical optimization of $Q$

Given some fixed set of basis function $\phi_0(s), \dots, \phi_N(s)$, we will search numerically for matrices $Q_0, \dots, Q_N$ such that the closed-loop transfer matrix $G_{zw}(s)$ satisfies given specifications when

$$
G_{zw}(s) = P_{zw}(s) - P_{zu}(s)Q(s)P_{yw}(s) \qquad \text{and } Q(s) = \sum_{k=0}^{N} Q_k \phi_k(s)
$$

Once $Q(s)$ has been determined, we will recover the desired controller from the formula

$$
C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1} Q(s)
$$

It is possible to choose the sequence $\phi_0(s), \phi_1(s), \phi_2(s), \dots$ such that every stable $Q$ can be approximated arbitrarily well. Hence, in principle, every convex control design problem can be solved this way.

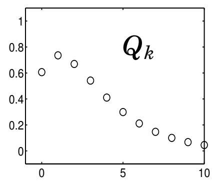
## Choice of basis functions

Many possibilities. Common choices:

- Laguerre basis polynomials,

$$
\phi_k(s) = \frac{1}{(s/a + 1)^k}
$$

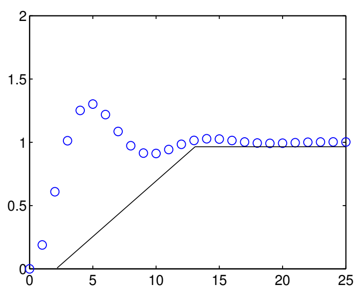
  where $a$ should be wisely selected
  (rule of thumb: close to bandwidth of closed-loop system)
- Pulse response parameterization (discrete time approximation)




## Specifications that lead to convex constraints

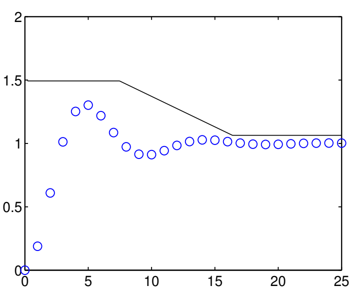
1. Stability of the closed-loop system
2. Lower bound on step response from $w_i$ to $z_j$ at time $t_i$
3. Upper bound on step response from $w_i$ to $z_j$ at time $t_i$
4. Upper bound on Bode amplitude from $w_i$ to $z_j$ at frequency $\omega_i$
5. Interval bound on Bode phase from $w_i$ to $z_j$ at frequency $\omega_i$

## Lower bound on step response



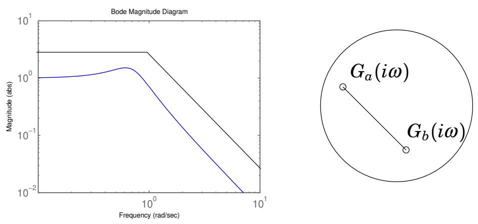

The step response depends linearly on $Q_k$, so every time $t_k$ with a lower bound gives a linear constraint.

## Upper bound on step response



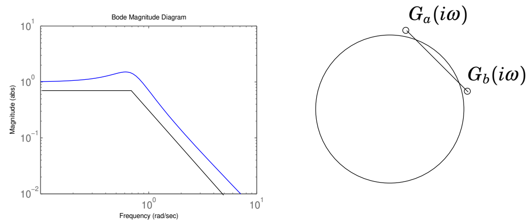

Every time $t_k$ with an upper bound also gives a linear constraint.

## Upper bound on Bode amplitude




An amplitude bound $|G(i\omega_i)| < c$ is a quadratic constraint.

## Lower bound on Bode amplitude




An lower bound $|G(i\omega_i)|$ is a **non-convex** quadratic constraint. This should be avoided in optimization.

## Synthesis by convex optimization

Quite general control synthesis problems can be stated as convex optimization problems in the variable $Q(s)$. The problem could have a quadratic objective, with linear/quadratic constraints, e.g.:

Minimize $\int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega) \overbrace{\sum_k Q_k \phi_k(i\omega)}^{Q(i\omega)} P_{yw}(i\omega)|^2 d\omega$ } quadratic objective

subj. to
step response $w_i \to z_j$ is smaller than $f_{ijk}$ at time $t_k$
step response $w_i \to z_j$ is bigger than $g_{ijk}$ at time $t_k$ } linear constraints
Bode magnitude $w_i \to z_j$ is smaller than $h_{ijk}$ at $\omega_k$ } quadratic constraints

Here $Q(s) = \sum_k Q_k \phi_k(s)$, where $\phi_1, \ldots, \phi_m$ are some fixed basis functions, and $Q_0, \ldots, Q_m$ are optimization variables.

Once $Q(s)$ has been determined, the controller is obtained as

$$C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1} Q(s)$$

## Software for convex optimization

- CVX – Matlab software for disciplined convex programming, developed at Stanford by Stephen Boyd and co-workers
  - Easily integrated with Python, Julia
  - CVXGEN – C code generation
- YALMIP – Matlab toolbox for convex and nonconvex optimization problems
- SeDuMi – software for optimiztion over symmetric cones
- SDPT3 – Matlab software for semidefinite programming
- ...

## Lecture 13 – Outline

Examples

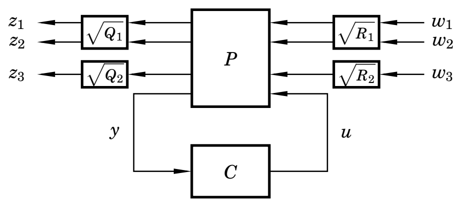
Introduction to convex optimization

Controller optimization using Youla parameterization

Examples revisited

## Example 1 (Doyle–Stein, 1979)

Reformulate LQG problem as extended plant model to be optimized:




Minimize

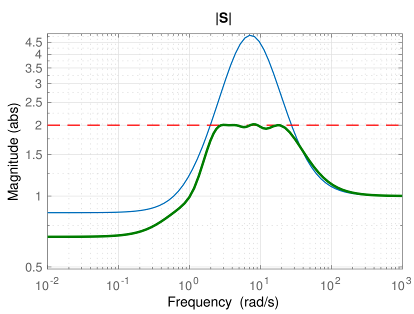
$$\int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega) \sum_k q_k \phi_k(i\omega) P_{yw}(i\omega)|^2 d\omega$$

with $q_k$ scalar and

$$\phi_k(s) = \frac{1}{(s/a+1)^k}$$

## Example 1 (Doyle–Stein, 1979)

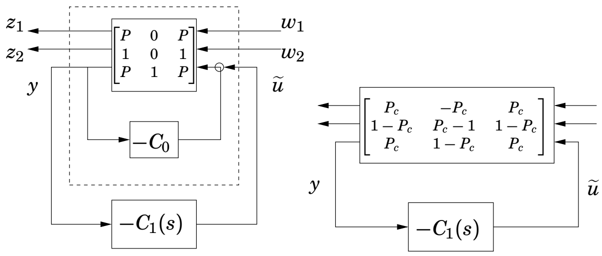
Green: Optimization-based design with constraint on $|S|$:




(Controller order: 12)

## Example 2 – DC-servo

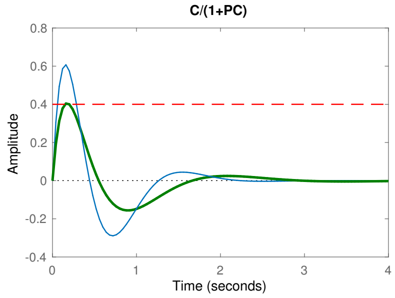
Introduce stabilizing controller $C_0$ and reformulate for optimization:




$$G_{zw}(s) = \begin{bmatrix} P_c & -P_c \\ 1-P_c & P_c - 1 \end{bmatrix} + \begin{bmatrix} P_c \\ 1-P_c \end{bmatrix} Q \begin{bmatrix} P_c & 1-P_c \end{bmatrix}$$

## Example 2 – DC-servo

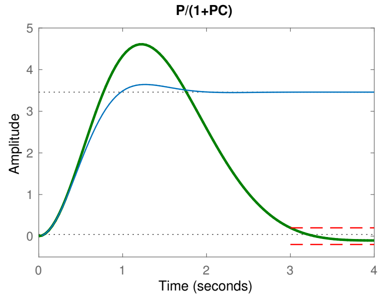
Green: Optimization with control signal limitation:




(Controller order: 14)

## Example 2 – DC-servo

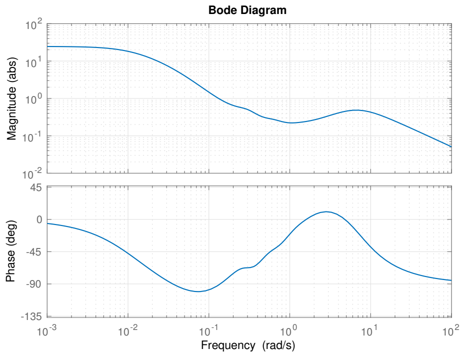
Green: Also adding the limit on $y$, $3 \leq t \leq 4$:




(Controller order: 14)

## Example 2 – DC-servo

Final controller:




Is it any good? With optimization, you get what you ask for!

## Summary

- There are efficient algorithms for convex optimization, e.g.
  - Linear programming (LP)
  - Quadratic programming (QP)
  - Second order cone programming (SOCP)
  - Semi-definite programming (SDP)
- The Youla parameterization allows us to use these algorithms for control synthesis
- Resulting controllers have high order. Order reduction will be studied in the next lecture.