FRTN10 Multivariable Control, Lecture 11

Automatic Control LTH, 2016

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

9. Linear-quadratic optimal control
10. Optimal observer-based feedback
11. **More on LQG**

L12-L14 Controller optimization: Numerical approach

Lecture 11 – Outline

1. Tuning the LQG design parameters
2. Robustness of LQG
3. Integral action, reference values

[Glad&Ljung sections 9.1–9.4 and 5.7]

Recall the main result of LQG

Given white noise (v_1, v_2) with intensity R and the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(k) \\ y(t) = Cx(t) + v_2(t) \end{cases} \qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\widehat{x}$ with $\frac{d}{dt}\widehat{x} = A\widehat{x} + Bu + K[y - C\widehat{x}]$. The stationary variance

$$\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$$

is minimized when

$$K = (PC^T + NR_{12})R_2^{-1} \qquad L = Q_2^{-1}(SB + Q_{12})^T$$

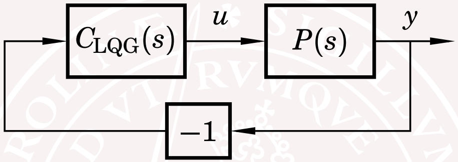
$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

$$0 = NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T$$

The minimal variance is

$$\mathrm{tr}(SNR_1N^T) + \mathrm{tr}[PL^T(B^TSB + Q_2)L]$$

The LQG controller



The controller transfer function (from $-y$ to u) is given by

$$C_{\mathrm{LQG}}(s) = L(sI - A + BL + KC)^{-1}K$$

- Same order as the (extended) plant model
- Strictly proper (if $R_2 > 0$)

Matlab: `reg(P,L,K)` or `lqgreg(Kest,L)`

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How to choose the cost function

- In rare instances, a quadratic cost function follows directly from the design specifications
- In most cases, Q_1, Q_2, Q_{12} must be tuned by the designer to achieve the desired closed-loop behavior

Starting point:

- Put $Q_{12} = 0$ and make Q_1, Q_2 diagonal
- Make the diagonal elements equal to the inverse value of the square of the allowed deviation:

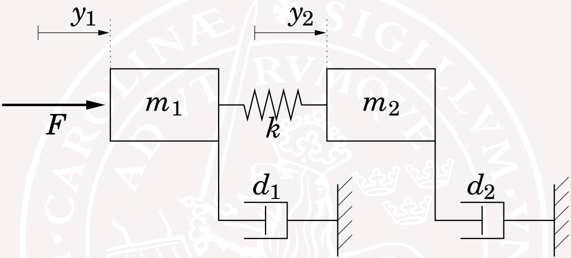
$$x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t)$$

$$= \left(\frac{x_1(t)}{x_1^{\max}}\right)^2 + \cdots + \left(\frac{x_n(t)}{x_n^{\max}}\right)^2 + \left(\frac{u_1(t)}{u_1^{\max}}\right)^2 + \cdots + \left(\frac{u_m(t)}{u_m^{\max}}\right)^2$$

How to tune the cost function

- To make the controller more aggressive, decrease Q_2 (or, equivalently, increase Q_1)
- To increase the damping of a state x_i, add penalty on $\dot{x}_i^2$ (may give cross-terms)
- To make a state x_i behave more like $\dot{x}_i = -\alpha x_i$, add penalty on $(\dot{x}_i + \alpha x_i)^2$ (gives cross-terms)

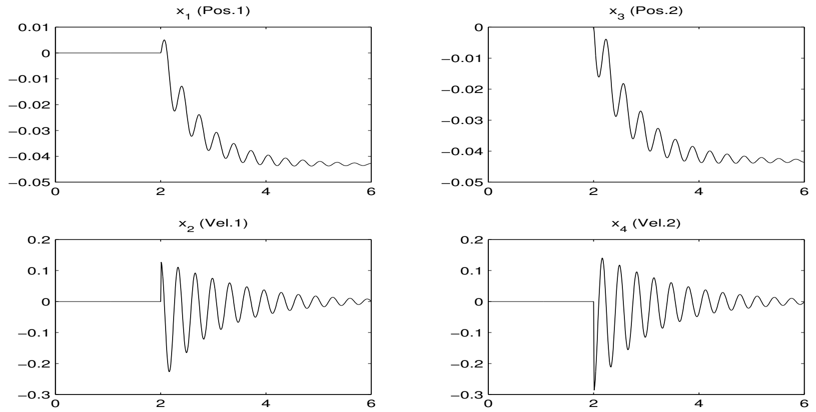
Example — Flexible servo



$$
\begin{aligned}
m_1 \frac{d^2 y_1}{dt^2} &= -d_1 \frac{dy_1}{dt} - k(y_1 - y_2) + F(t) \\
m_2 \frac{d^2 y_2}{dt^2} &= -d_2 \frac{dy_2}{dt} + k(y_1 - y_2)
\end{aligned}
$$

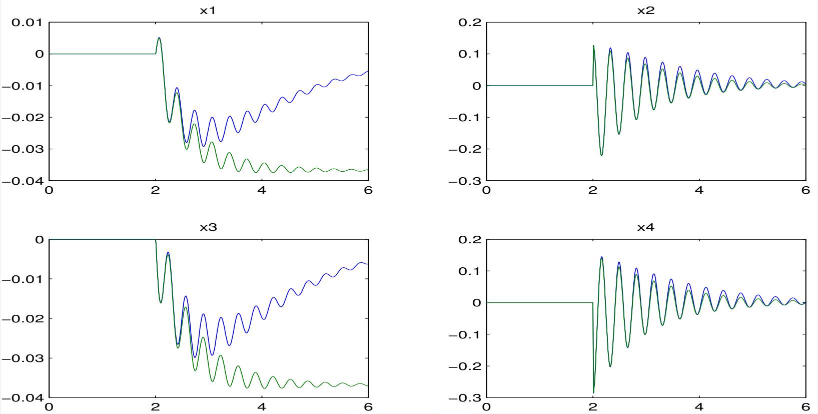
Introduce state variables $x_1 = y_1,\ x_2 = \dot{y}_1,\ x_3 = y_2,\ x_4 = \dot{y}_2$

Open loop response



Penalize velocity error or position error?

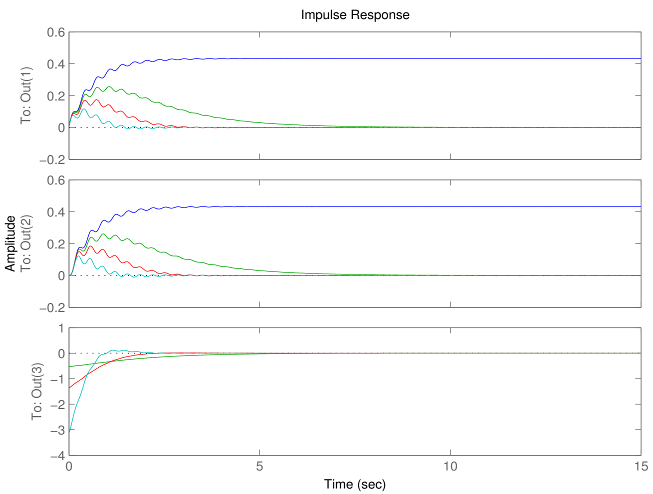
Minimize $\quad \mathbf{E}[x_2(t)^2 + x_4(t)^2 + u(t)^2]$ or $\mathbf{E}[x_1(t)^2 + x_3(t)^2 + u(t)^2]$?



When only velocity is penalized, a static position error remains

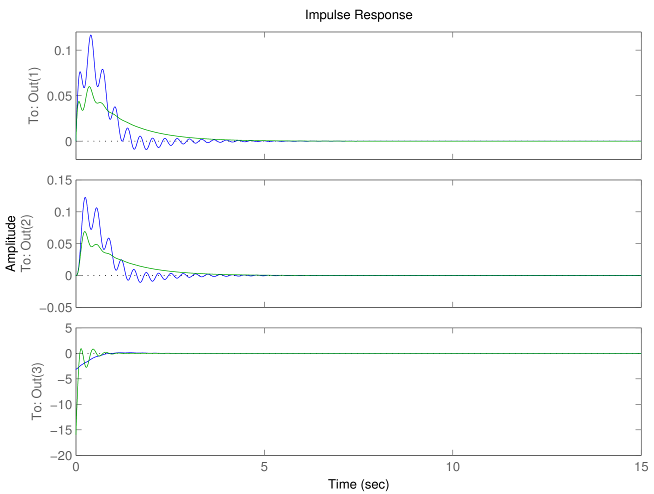
Position error control

Response of $x_1(t), x_3(t), u(t) = -Lx(t)$ to impulse disturbance.
$Q_1 = \text{diag}\{q, 0, q, 0\}$ ($q = 0, 1, 10, 100$), $Q_{12} = 0$, $Q_2 = 1$.
Large $q \Rightarrow$ fast response but large control signal.



Position+velocity error control

To reduce oscillations, penalize also velocity error. Comparision between $Q_1 = \text{diag}\{100, 0, 100, 0\}$ and $Q_1 = \text{diag}\{100, 100, 100, 100\}$.



How to choose/tune the noise intensities

- If no accurate noise characterization is available, as a starting point put $N = B$, $R_1 = I$, and $R_2 = \rho I$, where ρ is a scalar
- To make the controller less sensitive to measurement noise, increase R_2 (also makes the controller less aggressive)

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Robustness of LQG controllers

Candidate for the best abstract ever:

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—There are none.

Introduction

Considerable attention has been given lately to the issue of robustness of linear–quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 dB gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counterexample, this note answers that question; there are none.

[*IEEE Transactions on Automatic Control*, 23:4, 1978]

Example (Doyle–Stein, 1979)

Nice minimum-phase SISO plant, no fundamental limitations:

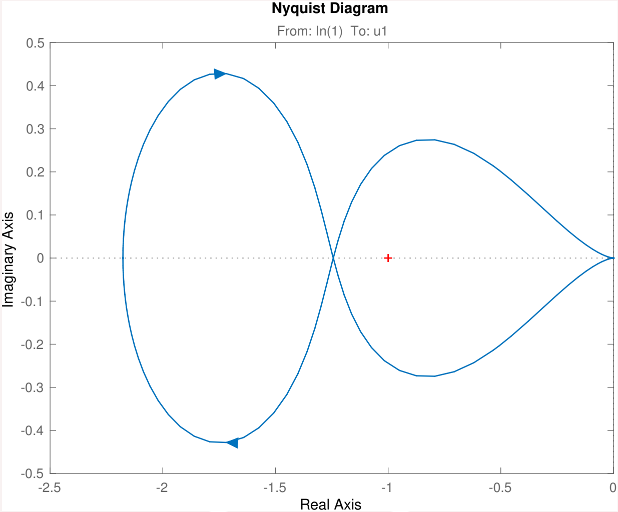
$$A = \begin{pmatrix} -4 & -3 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad N = \begin{pmatrix} -61 \\ 35 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$Q_1 = 80 \begin{pmatrix} 1 & \sqrt{35} \\ \sqrt{35} & 35 \end{pmatrix}, \quad Q_2 = 1, \quad R_1 = 1, \quad R_2 = 1$$

gives

- Control poles: $-7 \pm 2i$
- Observer poles: $-7.02 \pm 1.95i$

Example (Doyle–Stein, 1979)



$$M_s = 4.8, \varphi_m = 14.8°$$

Loop transfer recovery

The robustness of an LQG controller can often be improved by either

- adding a penalty proportional to C^TC to Q_1
- adding a penalty proportional to BB^T to NR_1N^T

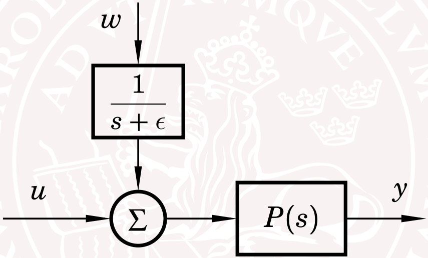
Makes the loop transfer function more similar to the state feedback (LQ) loop gain

Price: Higher controller gain

Lecture 11 – Outline

Integral action via disturbance modeling

Extend the plant model with a low-frequency disturbance acting on the process input:



The Kalman filter will include a (near) integrator

Easy to generalize to other disturbance models

Integral action via explicit integration

Add explicit integrators on the plant outputs

Gives extended plant model

$$\begin{pmatrix} \dot{x} \\ \dot{x}_i \end{pmatrix} = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ x_i \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N \\ 0 \end{pmatrix} v_1$$

Extended state feedback law from LQ design:

$$u = -\begin{pmatrix} L & L_i \end{pmatrix} \begin{pmatrix} x \\ x_i \end{pmatrix}$$

Including a penalty on x_i in the LQ design makes $x \to 0$ in case of a constant input load disturbance

Handling reference values

Simple solution:

$$u(t) = -L\hat{x}(t) + L_r r(t)$$

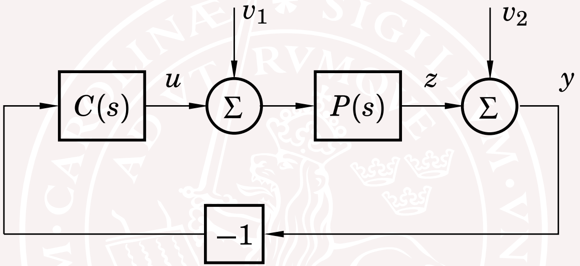
Assuming $z = Mx$ and $\dim z = \dim u$, select

$$L_r = [M(BL - A)^{-1}B]^{-1}$$

to ensure static gain I from r to z (in absence of disturbances)

A reference filter to further shape $G_{yr}(s)$ can be added if needed

LQG example — DC-servo



Process: $P(s) = \dfrac{20}{s(s+1)}$

Cost function: $J = \mathbf{E}\,(z^2 + u^2)$

White noise intensities: $R_1 = 1, R_2 = 1, R_{12} = 0$

LQG design

State-space model:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \overbrace{\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \overbrace{\begin{bmatrix} 20 \\ 0 \end{bmatrix}}^{B} u + \overbrace{\begin{bmatrix} 20 \\ 0 \end{bmatrix}}^{N} v_1$$

$$y = x_2 + v_2 \qquad z = x_2$$

Cost matrices:

$$Q_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \qquad Q_2 = 1$$

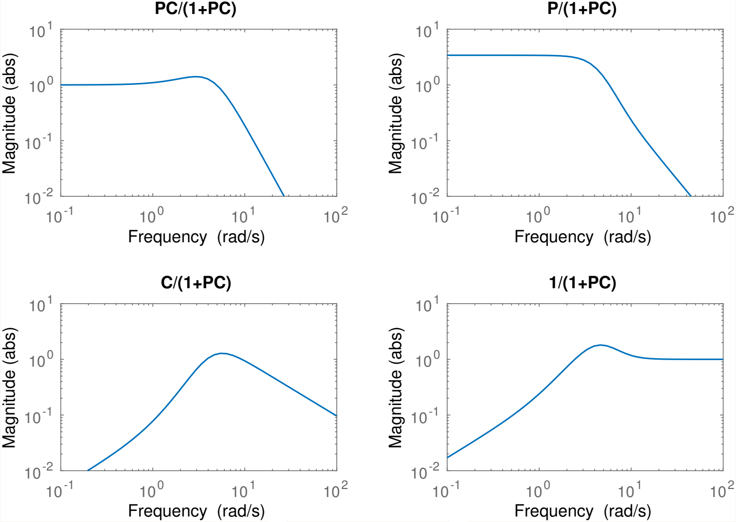
Solving the Riccati equations gives the optimal controller

$$\frac{d}{dt}\widehat{x} = (A - BL)\widehat{x} + K[y - C\widehat{x}] \qquad u = -L\widehat{x}$$

where

$$L = \begin{bmatrix} 0.2702 & 0.7298 \end{bmatrix} \qquad K = \begin{bmatrix} 20.0000 \\ 5.4031 \end{bmatrix}$$

Gang of four



Nonzero static gain in $\frac{P}{1+PC}$ indicates poor disturbance rejection

Integral action

Add explicit integrator on the output and extend the model:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_i\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 0\\ 0 & 1 & 0\end{bmatrix}}^{A_\text{e}} \begin{bmatrix}x_1\\ x_2\\ x_i\end{bmatrix} + \overbrace{\begin{bmatrix}20\\ 0\\ 0\end{bmatrix}}^{B_\text{e}} u + \overbrace{\begin{bmatrix}20\\ 0\\ 0\end{bmatrix}}^{N_\text{e}} v_1$$

$$y = x_2 + v_2, \quad z_1 = x_2, \quad z_2 = x_i$$

Minimization of $x_2^2 + 0.01x_i^2 + u^2$ gives the optimal state feedback

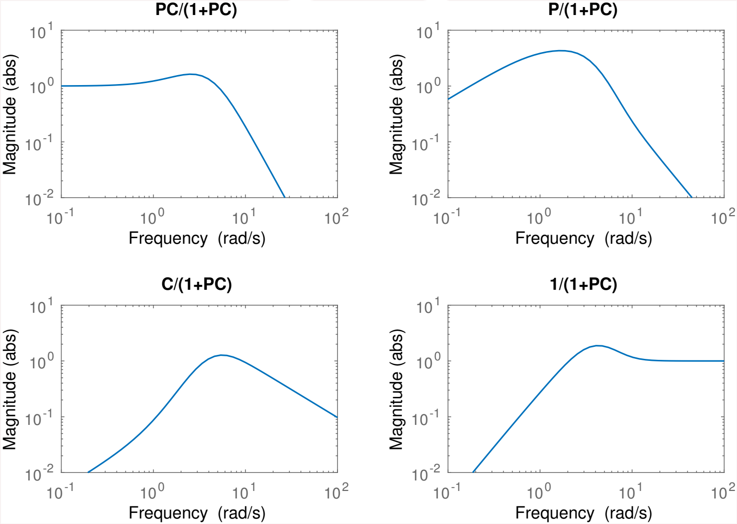
$$u = -L_e \begin{bmatrix}\widehat{x} & x_i\end{bmatrix}$$

where

$$L_\text{e} = \begin{bmatrix}0.2751 & 0.7569 & -0.1\end{bmatrix}$$

We use the same Kalman filter as before

Gang of four with integral action



Summary of LQG

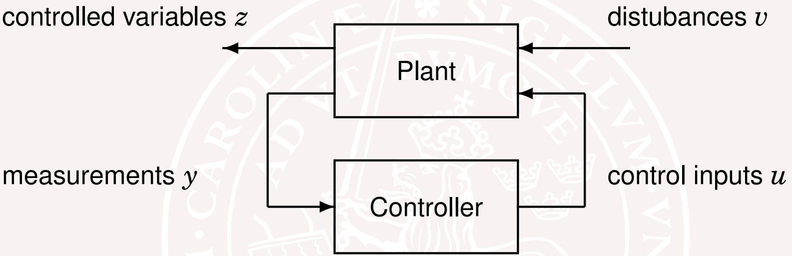
Advantages

- Works fine with multivariable models
- Observer structure ties to reality
- Always stabilizing
- Well developed theory, analytic solutions

Disadvantages

- High-order controllers (same order as the extended plant model)
- Sometimes hard to choose weights
- No robustness guarantees – must always check the resulting controller!

Alternative norms for optimization



H_∞ optimal control:

$$\text{Minimize} \quad \max_{\omega} \|G_{zv}(i\omega)\|$$

Can be solved using a couple of Riccati equations, similar to the LQG problem