FRTN10 Multivariable Control, Lecture 10

Automatic Control LTH, 2016

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

9. Linear-quadratic optimal control
10. **Optimal observer-based feedback**
11. More on LQG

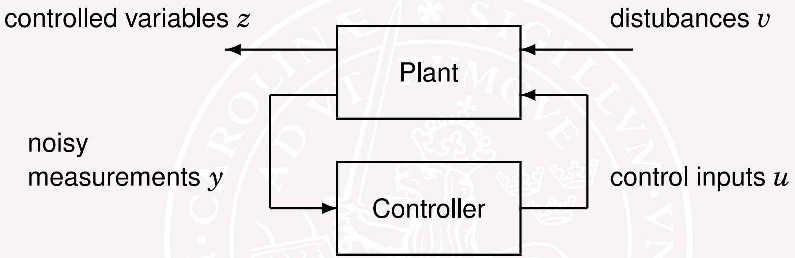
L12-L14 Controller optimization: Numerical approach

Lecture 10 – Outline

1. Observer-based feedback
2. The Kalman filter
3. LQG

[Glad&Ljung sections 9.1–9.4 and 5.7]

Goal: Linear-quadratic-Gaussian control (LQG)



For a linear plant, let v be white noise of intensity R. Find a controller that minimizes the output variance:

$$\mathbf{E}\,|z|^2 = \text{ trace} \int_{-\infty}^{\infty} RG_{zv}(i\omega)^* QG_{zv}(i\omega)d\omega$$

Previous lecture: State feedback solution.

Lecture 9 – Outline

1. Observer-based feedback
2. The Kalman filter
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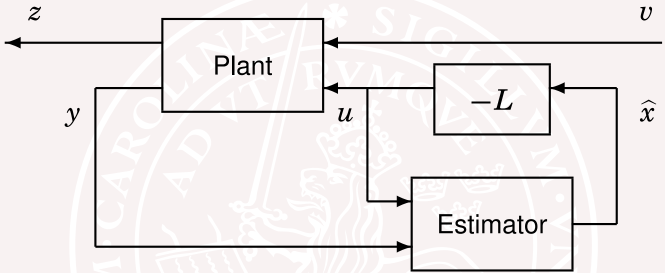
Output feedback using a state estimator



Plant:
$$\begin{cases} \frac{dx(t)}{dt} = Ax(t) + Bu(t) + Nv_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$$

Controller:
$$\begin{cases} \frac{d\hat{x}(t)}{dt} = A\widehat{x}(t) + Bu(t) + K[y(t) - C\widehat{x}(t)] \\ u(t) = -L\widehat{x}(t) \end{cases}$$

Closed-loop dynamics

Eliminate u and y:

$$\frac{dx(t)}{dt} = Ax(t) - BL\widehat{x}(t) + Nv_1(t)$$
$$\frac{d\hat{x}(t)}{dt} = A\widehat{x}(t) - BL\widehat{x}(t) + K[Cx(t) - C\widehat{x}(t)] + Kv_2(t)$$

Introduce the estimation error $\widetilde{x} = x - \widehat{x}$

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \widetilde{x}(t) \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(k) \\ \widetilde{x}(k) \end{bmatrix} + \begin{bmatrix} Nv_1(t) \\ Nv_1(t) - Kv_2(t) \end{bmatrix}$$

Two kinds of closed-loop poles

Control poles: $0 = \det(sI - A + BL)$

Observer poles: $0 = \det(sI - A + KC)$

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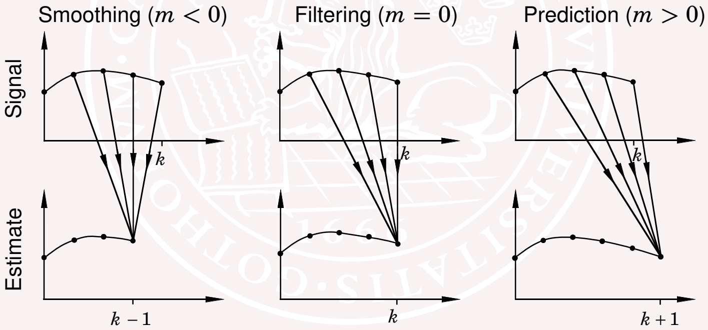
Rudolf E. Kálmán, 1930–2016

Recipient of the 2008 Charles Stark Draper Prize from the US National Academy of Engineering “for the devlopment and dissemination of the optimal digital technique (known as the Kalman Filter) that is pervasively used to control a vast array of consumer, health, commercial and defense products.”

Optimal filtering and prediction

- Wiener (1949): Stationary input-output formulation
- Kalman and Bucy (1961): Time-varying state-space formulation

General problem: Estimate $x(k+m)$ given $\{y(i), u(i) \,|\, i \le k\}$



Examples

Smoothing To estimate the Wednesday temperature based on temperature measurements from Tuesday, Wednesday and Thursday

Filtering To estimate the Wednesday temperature based on temperature measurements from Monday, Tuesday and Wednesday

Prediction To predict the Wednesday temperature based on temperature measurements from Sunday, Monday and Tuesday

The Kalman filter optimization problem

The estimation error dynamics are given by

$$\frac{d\widetilde{x}}{dt} = (A - KC)\widetilde{x} + \begin{pmatrix} N & -K \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

with

$$\mathbf{E} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^T = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} > 0$$

Optimization problem: Assuming that the system is observable, find the gain K that minimizes the stationary estimation error covariance

$$P = \mathbf{E}\, \widetilde{x}\widetilde{x}^T$$

Finding the optimal observer gain

The stationary error covariance P is given by the Lyapunov equation

$$(A-KC)P + P(A-KC)^T + \begin{pmatrix} N & -K \end{pmatrix} \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} \begin{pmatrix} N^T \\ -K^T \end{pmatrix} = 0$$

By completing the square,

$$AP+PA^T+NR_1N^T+(KR_2-PC^T-NR_{12})R_2^{-1}(KR_2-PC^T-NR_{12})^T \\ -(PC^T+NR_{12})R_2(PC^T+NR_{12}) = 0$$

we find that the minimium variance is attained for

$$K = (PC^T + NR_{12})R_2^{-1}$$

What remains is an algebraic Riccati equation,

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

The Kalman filter

[G&L Theorem 5.4]

Given an observable linear plant disturbed by white noise,

$$\begin{cases} \dot{x} = Ax + Bu + Nv_1 \\ y = Cx + v_2 \end{cases} \qquad \mathbf{E} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}^T = \begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} > 0$$

the optimal state estimator is given by the Kalman filter

$$\frac{d\hat{x}}{dt} = A\hat{x} + Bu + K(y - C\hat{x})$$

where K is given by

$$K = (PC^T + NR_{12})R_2^{-1}$$

where $P = \mathbf{E}\,(x - \hat{x})(x - \hat{x})^T > 0$ is the solution to

$$AP + PA^T + NR_1N^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T = 0$$

Remarks

Note that the optimal observer gain does not depend on what state(s) we are interested in. The Kalman filter produces the optimal estimate of **all states** at the same time.

The optimal observer gain K is static since we are solving a steady-state problem.

(The Kalman filter can also be derived for finite-horizon problems and problems with time-varying system matrices. We then obtain a Riccati differential equation for $P(t)$ and a time-varying filter gain $K(t)$)

Duality between state feedback and state estimation

State feedback	State estimation
A	A^T
B	C^T
Q_1	NR_1N^T
Q_2	R_2
Q_{12}	NR_{12}
S	P
L	K^T

Kalman filter in Matlab (1)

```
lqe  Kalman estimator design for continuous-time systems.

   Given the system
       .
       x = Ax + Bu + Gw            {State equation}
       y = Cx + Du + v             {Measurements}

   with unbiased process noise w and measurement noise v with
   covariances

       E{ww'} = Q,    E{vv'} = R,    E{wv'} = N ,

   [L,P,E] = lqe(A,G,C,Q,R,N)  returns the observer gain matrix L
   such that the stationary Kalman filter
       .
       x_e = Ax_e + Bu + L(y - Cx_e - Du)

   produces an optimal state estimate x_e of x using the sensor
   measurements y.  The resulting Kalman estimator can be formed
   with ESTIM.
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Kalman filter in Matlab (2)

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kalman  Kalman state estimator.

   [KEST,L,P] = kalman(SYS,QN,RN,NN) designs a Kalman estimator KEST for
   the continuous- or discrete-time plant SYS. For continuous-time plants
      .
      x = Ax + Bu + Gw            {State equation}
      y = Cx + Du + Hw + v        {Measurements}

   with known inputs u, process disturbances w, and measurement noise v,
   KEST uses [u(t);y(t)] to generate optimal estimates y_e(t),x_e(t) of
   y(t),x(t) by:
       .
      x_e  = Ax_e + Bu + L (y - Cx_e - Du)

      |y_e| = | C | x_e + | D | u
      |x_e|   | I |       | 0 |

   kalman takes the state-space model SYS=SS(A,[B G],C,[D H]) and the
   covariance matrices:

      QN = E{ww'},     RN = E{vv'},     NN = E{wv'}.
```

Example 1 – Kalman filter for an integrator

$\dot{x}(t) = v_1(t)$ — v_1 is white noise with intensity R_1

$y(t) = x(t) + v_2(t)$ — v_2 is white noise with intensity R_2

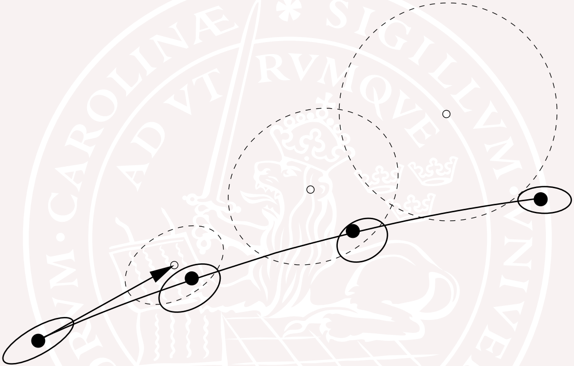
$$\frac{d\widehat{x}}{dt} = A\widehat{x}(t) + Bu(t) + K[y(t) - C\widehat{x}(t)]$$

Riccati equation $\quad 0 = R_1 - P^2/R_2 \quad \Rightarrow \quad P = \sqrt{R_1 R_2}$

Filter gain $\quad K = P/R_2 = \sqrt{R_1/R_2}$

Interpretation?

Example 2 – Tracking of a moving object



Dotted ellipses show estimates based on only a model with known initial state. Solid ellipses show Kalman filter estimates based on noisy measurements.

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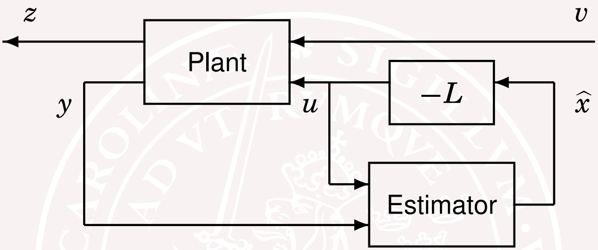
Optimal output feedback – LQG



Plant: $$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$$

Controller: $$\begin{cases} \frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) + Bu(t) + K[y(t) - C\widehat{x}(t)] \\ u(t) = -L\widehat{x}(t) \end{cases}$$

Minimize $$\mathbf{E}\,|z|^2 = \mathbf{E}\left(x^TQ_1x + 2x^TQ_{12}u + u^TQ_2u\right)$$

when v is white noise of intensity R

The separation principle

The following separation principle holds for linear systems with quadratic cost and Gaussian white noise disturbances:

- The optimal state feedback gain L is independent of the state uncertainty
- The optimal Kalman filter gain K is independent of the control objective

This makes it possible to optimize the state feedback law and the estimator separately.

LQG control

Given white noise (v_1, v_2) with intensity R and the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(k) \\ y(t) = Cx(t) + v_2(t) \end{cases} \qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\widehat{x}$ with $\frac{d}{dt}\widehat{x} = A\widehat{x} + Bu + K[y - C\widehat{x}]$. The stationary variance

$$\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$$

is minimized when

$$K = (PC^T + NR_{12})R_2^{-1} \qquad L = Q_2^{-1}(SB + Q_{12})^T$$

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

$$0 = NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T$$

The minimal variance is

$$\mathrm{tr}(SNR_1N^T) + \mathrm{tr}[PL^T(B^TSB + Q_2)L]$$

Example – LQG control of an integrator

Consider the problem to minimize $\mathbf{E}(Q_1 x^2 + Q_2 u^2)$ for

$$\begin{cases} \dot{x}(t) = u(t) + v_1(t) \\ y(t) = x(t) + v_2(t) \end{cases} \qquad R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

The observer-based controller

$$\begin{cases} \frac{d}{dt}\widehat{x}(t) = A\widehat{x}(t) + Bu(t) + K[y(t) - C\widehat{x}(t)] \\ u(t) = -L\widehat{x}(t) \end{cases}$$

is optimal for K and L computed as follows:

$$0 = Q_1 - S^2/Q_2 \quad \Rightarrow \quad S = \sqrt{Q_1 Q_2} \quad \Rightarrow \quad L = S/Q_2 = \sqrt{Q_1/Q_2}$$

$$0 = R_1 - P^2/R_2 \quad \Rightarrow \quad P = \sqrt{R_1 R_2} \quad \Rightarrow \quad K = P/R_2 = \sqrt{R_1/R_2}$$

Example – Control of a LEGO segway

Essentially an inverted pendulum – classical control problem

Sensors: Accelerometer, gyroscope

Actuators: DC motors

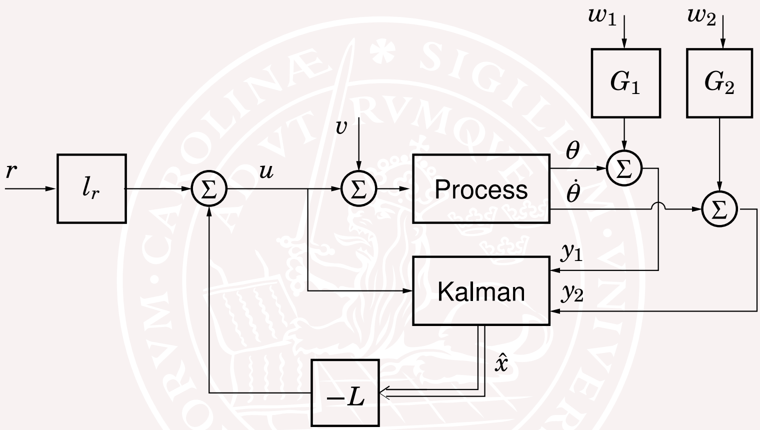
Sensor fusion

The two sensors have very different characteristics:

- The accelerometer is good for measuring the steady-state angle but very sensitive to disturbances at higher frequencies
- The gyroscope can measure the angular speed and track fast movements, but due to drift it cannot track the steady-state angle

Solution: Sensor fusion using Kalman filter

Modeling for LQG design



- v, w_1, w_2 – white noise sources
- $G_1(s) = \frac{s+a}{s/N+a}$ – models the inaccuary of the accelerometer
- $G_2(s) = \frac{s+b}{s}$ – models the incaccuary of the gyroscope

Next lecture: More on LQG

- Robustness of LQG?
- How to choose the design weights Q and R?
- How to handle reference signals and integral action?
- Examples