FRTN10 Multivariable Control, Lecture 9

Automatic Control LTH, 2016

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

9. **Linear-quadratic optimal control**
10. Optimal observer-based feedback
11. More on LQG

L12-L14 Controller optimization: Numerical approach

Lecture 9 – Outline

1. Dynamic programming
2. The Riccati equation
3. Optimal state feedback
4. Stability and robustness

Sections 9.1–9.4 + 5.7 in the book treat essentially the same material as we cover in lectures 9–11. However, the main derivation of the LQG controller in 9.A and 18.5 is different.

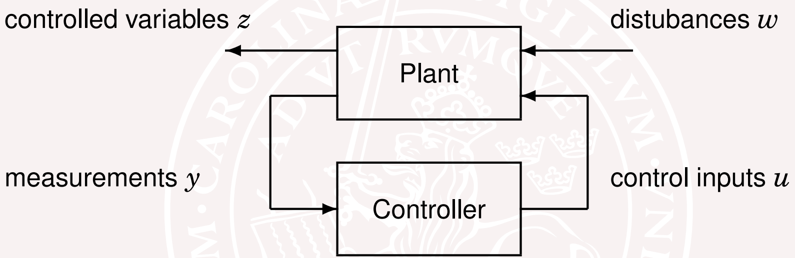
Math repetition

Suppose the matrix Q is symmetric: $Q = Q^T$. Then

- $Q > 0$ means that $x^T Q x > 0$ for any $x \neq 0$
  - True iff all eigenvalues of Q are positive.
  - We say that Q is positive definite.

- $Q \geq 0$ means that $x^T Q x \geq 0$ for any $x \neq 0$
  - True iff all eigenvalues of Q are non-negative.
  - We say that Q is positive semidefinite.

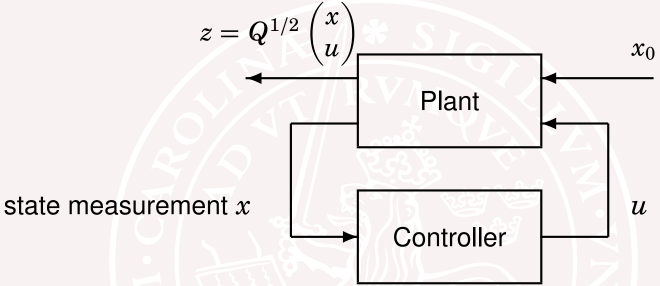
A general optimization setup



The objective is to find a controller that optimizes the transfer matrix $G_{zw}(s)$ from disturbances (and setpoints) w to controlled outputs z.

Lectures 9–11: Problems with analytic solutions
Lectures 12–14: Problems with numeric solutions

Today's problem: Optimal state feedback



Minimize $\displaystyle\int_0^\infty \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} dt$

subject to $\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$

Why linear-quadratic optimal control?

- Analytic solution
- Always stabilizing
- Works for MIMO systems
- Guaranteed robustness (in the state feedback case)
- Foundation for more advanced methods like model-predictive control (MPC)

Lecture 9 – Outline

1. Dynamic programming
2. The Riccati equation
3. Optimal state feedback
4. Stability and robustness

Mini-problem

Determine u_0 and u_1 if the objective is to minimize

$$x_1^2 + x_2^2 + u_0^2 + u_1^2$$

when

$$x_1 = x_0 + u_0$$
$$x_2 = x_1 + u_1$$

Hint: Go backwards in time.

Solution (1) to mini-problem

$$f(u_0, u_1) = x_1^2 + x_2^2 + u_0^2 + u_1^2 = (\underbrace{x_0 + u_0}_{x_1})^2 + (\underbrace{(x_0 + u_0)}_{x_1} + u_1)^2 + u_0^2 + u_1^2$$

$$= 2{x_0}^2 + (2u_1 + 4u_0)\, x_0 + 2u_0u_1 + 2{u_1}^2 + 3{u_0}^2$$

$$\frac{\partial f}{\partial u_0} = 4x_0 + 2u_1 + 6u_0 = 0$$
$$\frac{\partial f}{\partial u_1} = 2x_0 + 2u_0 + 4u_1 = 0$$

(Don't forget to check whether maximum or minimum...)

$$\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} -4x_0 \\ -2x_0 \end{bmatrix} \Longrightarrow \begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = \begin{bmatrix} -\frac{3}{5}x_0 \\ -\frac{1}{5}x_0 \end{bmatrix} \Longrightarrow f_{\min} = \frac{3}{5}x_0^2$$

This sequence depends on the initial value x_0 only (no feedback). Unwieldy for larger problems. For robustness and computational reasons it is better to have a feedback solution!

Solution (2) to mini-problem

Better solution: Break the problem into smaller parts that can be solved sequentially:

$$\min_{u_0,u_1}\left\{x_1^2+x_2^2+u_0^2+u_1^2\right\}=\min_{u_0}\left\{x_1^2+u_0^2+\underbrace{\min_{u_1}\left\{x_2^2+u_1^2\right\}(x_1)}_{J_1(x_1)}\right\}$$

$$J_1(x_1)=\min_{u_1}\left\{(x_1+u_1)^2+u_1^2\right\}=\dots$$

Gives

$$u_0=-\tfrac{3}{5}x_0$$
$$u_1=-\tfrac{1}{2}x_1$$

Quadratic optimal cost

It can be shown that the optimal cost on the time interval $[t, \infty)$ is quadratic:

$$\min_{u[t,\infty)} \int_t^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T Q \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} d\tau = x^T(t)Sx(t)$$

when

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} > 0$$

Dynamic programming, Richard E. Bellman, 1957

An optimal trajectory on the time interval $[t, T]$ must be optimal also on each of the subintervals $[t, t+\epsilon]$ and $[t+\epsilon, T]$.

Dynamic programming in linear-quadratic control

Let $x_t = x(t)$, $u_t = u(t)$. For a time step of length ϵ,

$$x(t+\epsilon) = x_t + (Ax_t + Bu_t)\epsilon \qquad \text{as } \epsilon \to 0$$

$$\begin{aligned}
x_t^T S x_t &= \min_{u[t,\infty)} \int_t^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T Q \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} d\tau \\
&= \min_{u[t,\infty)} \left\{ \begin{pmatrix} x_t \\ u_t \end{pmatrix}^T Q \begin{pmatrix} x_t \\ u_t \end{pmatrix} \epsilon + \int_{t+\epsilon}^\infty \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix}^T Q \begin{pmatrix} x(\tau) \\ u(\tau) \end{pmatrix} d\tau \right\} \\
&= \min_{u_t} \Big\{ (x_t^T Q_1 x_t + 2x_t^T Q_{12} u_t + u_t^T Q_2 u_t)\epsilon \\
&\qquad + \Big[x_t + (Ax_t + Bu_t)\epsilon\Big]^T S \Big[x_t + (Ax_t + Bu_t)\epsilon\Big] \Big\}
\end{aligned}$$

by definition of S. Neglecting ϵ^2 gives **Bellman's equation**:

$$0 = \min_{u_t} \left\{ \left(x_t^T Q_1 x_t + 2x_t^T Q_{12} u_t + u_t^T Q_2 u_t \right) + 2x_t^T S (Ax_t + Bu_t) \right\}$$

Lecture 9 – Outline

1. Dynamic programming
2. The Riccati equation
3. Optimal state feedback
4. Stability and robustness

Completion of squares

Suppose $Q_u > 0$. Then

$$x^T Q_x x + 2x^T Q_{xu} u + u^T Q_u u$$
$$= (u + Q_u^{-1} Q_{xu}^T x)^T Q_u (u + Q_u^{-1} Q_{xu}^T x) + x^T (Q_x - Q_{xu} Q_u^{-1} Q_{xu}^T) x$$

is minimized by

$$u = -Q_u^{-1} Q_{xu}^T x$$

The minimum is

$$x^T (Q_x - Q_{xu} Q_u^{-1} Q_{xu}^T) x$$

The Riccati equation

Completion of squares in Bellman's equation gives

$$
\begin{aligned}
0 &= \min_{u_t} \left\{ \left(x_t^T Q_1 x_t + 2x_t^T Q_{12} u_t + u_t^T Q_2 u_t \right) + 2x_t^T S(Ax_t + Bu_t) \right\} \\
&= \min_{u_t} \left\{ x_t^T [Q_1 + A^T S + SA] x_t + 2x_t^T [Q_{12} + SB] u_t + u_t^T Q_2 u_t \right\} \\
&= x_t^T \left(Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T \right) x_t
\end{aligned}
$$

with minimum attained for

$$u_t = -Q_2^{-1}(SB + Q_{12})^T x_t$$

The equation

$$0 = Q_1 + A^T S + SA - (SB + Q_{12}) Q_2^{-1} (SB + Q_{12})^T$$

is called the *algebraic Riccati equation*

Jocopo Francesco Riccati, 1676–1754

Solving algebraic Riccati equations in Matlab

```
care  Solve continuous-time algebraic Riccati equations.

   [X,L,G] = care(A,B,Q,R,S,E) computes the unique stabilizing
   solution X of the continuous-time algebraic Riccati equation
                                          -1
     A'XE + E'XA - (E'XB + S)R  (B'XE + S') + Q = 0 .

   When omitted, R, S and E are set to the default values R=I,
   S=0, and E=I.  Beside the solution X, care also returns the
   gain matrix
                   -1
          G = R  (B'XE + S')

   and the vector L of closed-loop eigenvalues (i.e.,
   EIG(A-B*G,E)).
```

Lecture 9 – Outline

1. Dynamic programming
2. The Riccati equation
3. Optimal state feedback
4. Stability and robustness

Linear-quadratic optimal control

Control problem:

Minimize $$\int_0^\infty \Big(x^T(t)Q_1x(t) + 2x^T(t)Q_{12}u(t) + u^T(t)Q_2u(t)\Big)dt$$

subject to $\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0$

Solution: Assume (A, B) controllable. Then there is a unique $S > 0$ solving the algebraic Riccati equation

$$0 = Q_1 + A^TS + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

The optimal control law is $u = -Lx$ with $L = Q_2^{-1}(SB + Q_{12})^T$.

The minimal cost is $x_0^TSx_0$.

Remarks

Note that the optimal control law does not depend on x_0.

The optimal feedback gain L is static since we are solving an infinite-horizon problem.

(LQ theory can also be applied to finite-horizon problems and problems with time-varying system matrices. We then obtain a Riccati differential equation for $S(t)$ and a time-varying state feedback, $u(t) = -L(t)x(t)$)

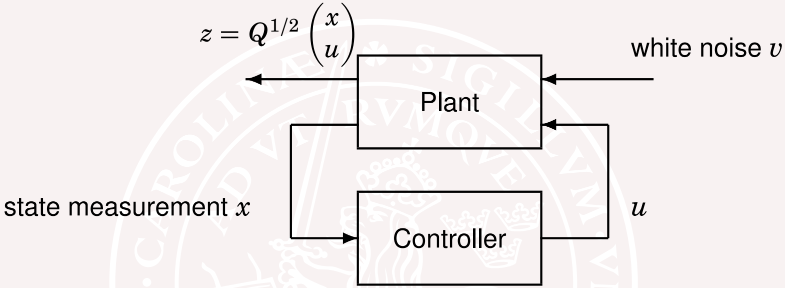
Example: Control of an integrator

For $\dot{x}(t) = u(t)$, $x(0) = x_0$,

Minimize	$J = \int_0^\infty \left\{ x(t)^2 + \rho u(t)^2 \right\} dt$
Riccati equation	$0 = 1 - S^2/\rho \quad \Rightarrow \quad S = \sqrt{\rho}$
Controller	$L = S/\rho = 1/\sqrt{\rho} \quad \Rightarrow \quad u = -x/\sqrt{\rho}$
Closed loop system	$\dot{x} = -x/\sqrt{\rho} \quad \Rightarrow \quad x = x_0 e^{-t/\sqrt{\rho}}$
Optimal cost	$J^* = x_0^T S x_0 = x_0^2 \sqrt{\rho}$

What values of ρ give the fastest response? Why?
What values of ρ give smallest optimal cost? Why?

Stochastic interpretation of LQ control



Minimize $\quad J = \mathbf{E}\left\{x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right\}$

subject to $\quad \dot{x}(t) = Ax(t) + Bu(t) + v(t)$

where v is white noise with intensity R. Same Riccati solution S as in the deterministic case. The optimal cost is

$$J^* = \mathrm{tr}\, SR$$

Lecture 9 – Outline

1. Dynamic programming
2. The Riccati equation
3. Optimal state feedback
4. Stability and robustness

Theorem: Stability of the closed-loop system

Assume that

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} > 0$$

and that there exists a solution $S > 0$ to the algebraic Riccati equation. Then the optimal controller $u(t) = -Lx(t)$ gives an asymptotically stable closed-loop system $\dot{x}(t) = (A - BL)x(t)$.

Proof:

$$\begin{aligned} \frac{d}{dt} x^T(t) S x(t) &= 2x^T S \dot{x} = 2x^T S(Ax + Bu) \\ &= -\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right) < 0 \text{ for } x(t) \neq 0 \end{aligned}$$

Hence $x^T(t) S x(t)$ is decreasing and tends to zero as $t \to \infty$.

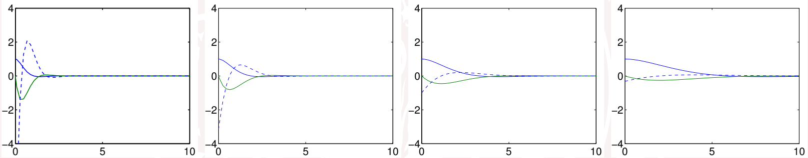
Solving the LQ problem in Matlab

```
lqr  Linear-quadratic regulator design for state space systems

 [K,S,E] = lqr(SYS,Q,R,N) calculates the optimal gain matrix K
 such that:

   * For a continuous-time state-space model SYS, the state-
     feedback law u = -Kx  minimizes the cost function

           J = Integral {x'Qx + u'Ru + 2*x'Nu} dt

     subject to the system dynamics  dx/dt = Ax + Bu

 The matrix N is set to zero when omitted. Also returned are
 the solution S of the associated algebraic Riccati equation
 and the closed-loop eigenvalues E = EIG(A-B*K).
```

Example – Double integrator

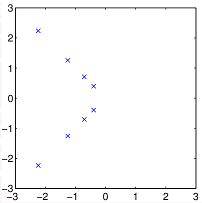
$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Q_2 = \rho \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

States and inputs (dotted) for $\rho = 0.01$, $\rho = 0.1$, $\rho = 1$, $\rho = 10$

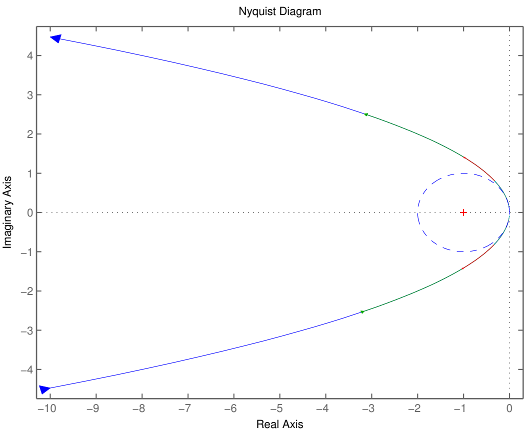


Closed loop poles:

$$s = 2^{-1/2}\rho^{-1/4}(-1 \pm i)$$



Robustness of optimal state feedback



The distance from the loop gain $L(i\omega I - A)^{-1}B$ to -1 is never smaller than 1. This is always true(!) for linear-quadratic optimal state feedback when $Q_1 > 0$, $Q_{12} = 0$ and $Q_2 = \rho > 0$ is scalar. Hence the phase margin is at least $60°$ and the gain margin is infinite!

Proof of robustness

Using the Riccati equation

$$0 = Q_1 + A^T S + SA - L^T Q_2 L, \qquad L = Q_2^{-1}(SB + Q_{12})^T$$

it is straightforward to verify (see [G&L Lemma 5.2]) that

$$\left[I + L(i\omega - A)^{-1}B\right]^* Q_2 \left[I + L(i\omega - A)^{-1}B\right] = \begin{bmatrix}(i\omega - A)^{-1}B \\ I\end{bmatrix}^* \begin{bmatrix}Q_1 & Q_{12} \\ Q_{12}^* & Q_2\end{bmatrix} \begin{bmatrix}(i\omega - A)^{-1}B \\ I\end{bmatrix}$$

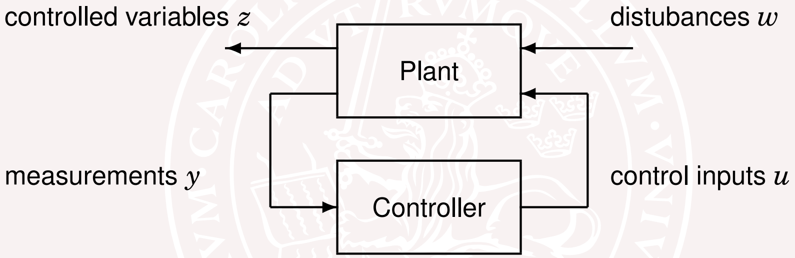
In particular, with $Q_1 > 0$, $Q_{12} = 0$, $Q_2 = \rho > 0$

$$\begin{aligned}\left[1 + L(i\omega - A)^{-1}B\right]^* \rho \left[1 + L(i\omega - A)^{-1}B\right] &= B^T[(i\omega - A)^{-1}]^* Q_1 (i\omega - A)^{-1}B + \rho \\ &\geq \rho\end{aligned}$$

Dividing by ρ gives

$$|1 + L(i\omega - A)^{-1}B|^2 \geq 1$$

Next lecture: Linear-quadratic-Gaussian control



For a linear plant, minimize a quadratic function of the map from disturbances w to controlled variables z