## FRTN10 Multivariable Control, Lecture 8

**Automatic Control LTH, 2016**

## Course Outline

- L1-L5 Specifications, models and loop-shaping by hand
- L6-L8 Limitations on achievable performance
  6. Controllability, observability, multivariable zeros
  7. Fundamental limitations
  8. **Multivariable and decentralized control**
- L9-L11 Controller optimization: Analytic approach
- L12-L14 Controller optimization: Numerical approach

## Lecture 8 – Outline

1. Transfer functions for MIMO systems
2. Limitations due to RHP zeros
3. Decentralized control
4. Decoupling

See “Lecture notes” and [G&L, Chapters 1, 7.7 (first part) and 8.3]

## Lecture 8 – Outline

Transfer functions for MIMO systems

Limitations due to RHP zeros

Decentralized control

Decoupling

## Typical process control system

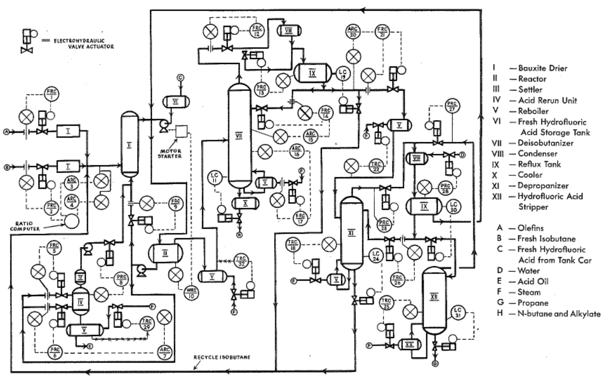
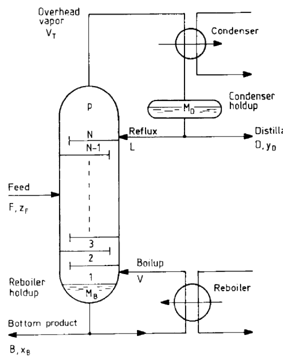



Figure 13–6. Automatic control system for Perco motor fuel alkylation process.

## Example system: Distillation column




Raw oil inserted at bottom $\rightarrow$ different petro-chemical subcomponents extracted

## Example system: Distillation column

Linear model:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{4}{50s+1}e^{-27s} & \frac{1.8}{60s+1}e^{-28s} & \frac{5.9}{50s+1}e^{-27s} \\ \frac{5.4}{50s+1}e^{-18s} & \frac{5.7}{60s+1}e^{-14s} & \frac{6.9}{40s+1}e^{-15s} \end{bmatrix}}_{P(s)} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$$

**Outputs:**

$y_1$ = top draw composition

$y_2$ = side draw composition

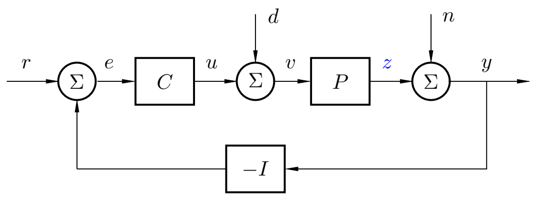
**Inputs:**

$u_1$ = top draw flowrate

$u_2$ = side draw flowrate

$u_3$ = bottom temperature control input

## Multivariable transfer functions




$P$ and $C$ are matrices – order matters!

$$Z(s) = PC \cdot R(s) + P \cdot D(s) - PC \cdot [N(s) + Z(s)]$$
$$[I + PC]Z(s) = PC \cdot R(s) + P \cdot D(s) - PC \cdot N(s)$$

$$Z(s) = [I + PC]^{-1} \cdot (PC \cdot R(s) + P \cdot D(s) - PC \cdot N(s))$$

Notice that $[I + PC]^{-1}$ is generally not the same as $[I + CP]^{-1}$.

## Sensitivity functions for MIMO systems

Output sensitivity function
$S = (I + PC)^{-1}$ $\qquad G_{? \to ?}$

Input sensitivity function
$(I + CP)^{-1}$ $\qquad G_{? \to ?}$

Complementary sensitivity function
$T = (I + PC)^{-1}PC$ $\qquad G_{? \to ?}$

Mini-problem:

Find the transfer functions above in the block diagram on the previous slide.

## Some useful math relations

Notice the following identities:

$$(i)\quad [I + PC]^{-1}P = P[I + CP]^{-1}$$

$$(ii)\quad C[I + PC]^{-1} = [I + CP]^{-1}C$$

$$(iii)\quad T = P[I + CP]^{-1}C = PC[I + PC]^{-1} = [I + PC]^{-1}PC$$

$$(iv)\quad S + T = I$$

Proof:
The first equality follows by multiplication on both sides with $(I + PC)$ from the left and with $(I + CP)$ from the right.

Left: $[I + PC][I + PC]^{-1}P[I + CP] = I \cdot [P + PCP] = [I + PC]P$
Right: $[I + PC]P[I + CP]^{-1}[I + CP] = [I + PC]P \cdot I = [I + PC]P$

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Transfer functions for MIMO systems

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Decoupling

## Hard limitations from RHP zeros

[G&L Theorem 7.9]

Assume that the MIMO system $P(s)$ has a transfer zero $z_i$ in the RHP.

Let $S(s) = [I + P(s)C(s)]^{-1}$ be the sensitivity function and let $W_S(s)$ be a scalar, stable and minimum phase transfer function. Then the specification

$$\|W_S S\|_\infty = \sup_\omega \bar{\sigma}(W_S(i\omega)S(i\omega)) \le 1$$

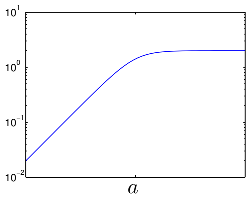
cannot be fulfilled unless

$$|W_S(z_i)| \le 1$$

## Example

Assume the specification $W_S(s) = \frac{s+a}{2s}$

$$|W_S(z_i)| = \frac{z_i + a}{2z_i} \le 1 \quad \Rightarrow \quad a \le z_i$$




## Non-minimum-phase MIMO system

Example [G&L, Ch 1]
Consider a feedback system $Y(s) = (I + PC)^{-1}PCR(s)$ with the multivariable process

$$P(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Computing the determinant

$$\det P(s) = \frac{2}{(s+1)^2} - \frac{3}{(s+2)(s+1)} = \frac{-s+1}{(s+1)^2(s+2)}$$

shows that the process has a RHP zero at $s = 1$, which will limit the achievable performance.

See lecture notes for details of the following slides (checking three different controllers)

## Example – Controller 1

The controller

$$C_1(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & -\frac{3K_2(s+0.5)}{s(s+2)} \\ -\frac{K_1(s+1)}{s} & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

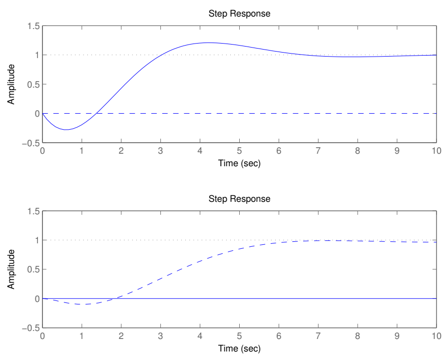
gives the diagonal loop transfer matrix

$$P(s)C_1(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & 0 \\ 0 & \frac{K_2(s+0.5)(-s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

Hence the system is decoupled into to scalar loops, each with an unstable zero at $s = 1$ that limits the bandwidth.

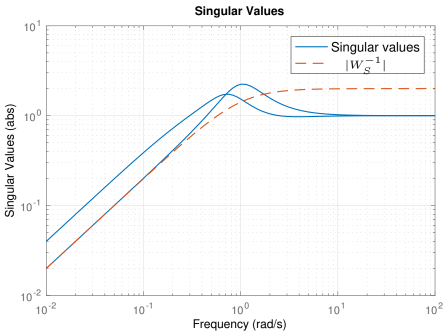
The closed-loop step responses for $K_1 = K_2 = 1$ are shown on next slide.

## Step responses using Controller 1




Closed-loop step responses with decoupling controller $C_1(s)$ for the two outputs $y_1$ (solid) and $y_2$ (dashed). The upper plot is for a reference step for $y_1$. The lower plot is for a reference step for $y_2$.

## Sensitivity sigma plot using Controller 1




$W_S(s) = \frac{s+1.001}{2s}$, impossible to meet

## Example – Controller 2

The controller

$$C_2(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & K_2 \\ -\frac{K_1(s+1)}{s} & K_2 \end{bmatrix}$$

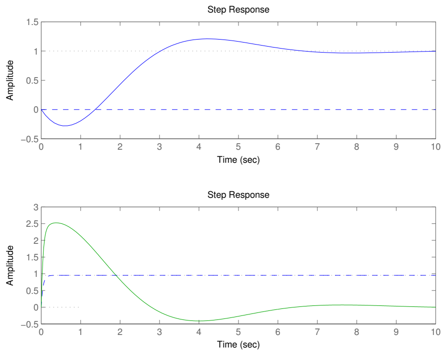
gives the triangular loop transfer matrix

$$P(s)C_2(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & \frac{K_2(5s+7)}{(s+2)(s+1)} \\ 0 & \frac{2K_2}{s+1} \end{bmatrix}$$

Now the decoupling is only partial:
Output $y_2$ is not affected by $r_1$. Moreover, there is no unstable zero that limits the rate of response in $y_2$!

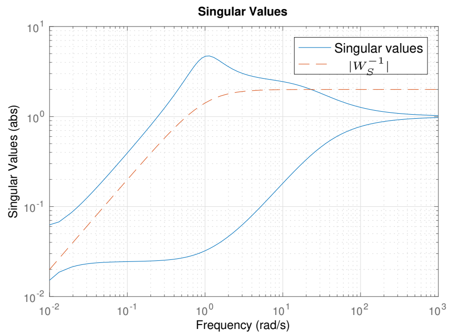
The closed loop step responses for $K_1 = 1, K_2 = 10$ are shown on next slide.

## Step responses using Controller 2




Closed-loop step responses with controller $C_2(s)$ for the two outputs $y_1$ (solid) and $y_2$ (dashed). The RHP zero does not prevent a fast $y_2$-response to $r_2$ but at the price of a simultaneous undesired response in $y_1$.

## Sensitivity sigma plot using Controller 2




$W_S(s) = \frac{s+1.001}{2s}$, impossible to meet

## Example – Controller 3

The controller

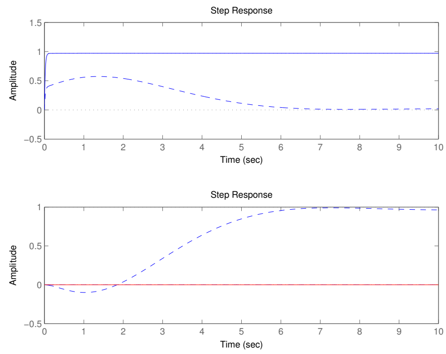
$$C_3(s) = \begin{bmatrix} K_1 & \frac{-3K_2(s+0.5)}{s(s+2)} \\ K_1 & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

gives the triangular loop transfer matrix

$$P(s)C_3(s) = \begin{bmatrix} \frac{K_1(5s+7)}{(s+1)(s+2)} & 0 \\ \frac{2K_1}{s+1} & \frac{K_2(-1+s)(s+0.5)}{s(s+1)^2(s+2)} \end{bmatrix}$$

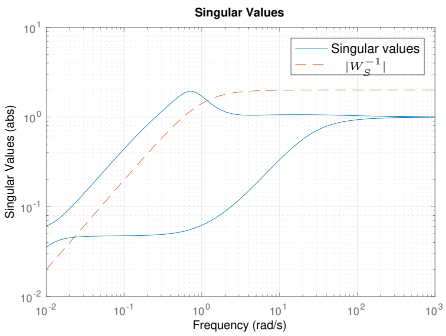
In this case $y_1$ is decoupled from $r_2$ and can respond arbitrarily fast for high values of $K_1$, at the expense of bad behavior in $y_2$. Step responses for $K_1 = 10, K_2 = 1$ are shown on next slide.

## Step responses using Controller 3




Closed-loop step responses with controller $C_3(s)$ for the two outputs $y_1$ (solid) and $y_2$ (dashed). The RHP zero does not prevent a fast $y_1$-response to $r_1$ but at the price of a simultaneous undesired response in $y_2$.

## Sensitivity sigma plot using Controller 3




$W_S(s) = \frac{s+1.001}{2s}$, impossible to meet

## Example – summary

To summarize, the example shows that even though a **multivariable RHP zero always gives a performance limitation**, it is **possible to influence** where the effects should show up.

## Lecture 8 – Outline

Transfer functions for MIMO systems

Limitations due to RHP zeros

Decentralized control

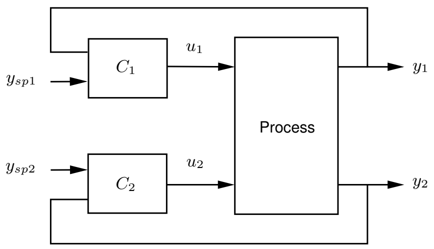
Decoupling

## Decentralized control

Background in process control:

- A few important variables were controlled using the simple loop paradigm: one sensor, one actuator, one controller
- As more loops were added, interaction was handled using feedforward, cascade and midrange control, selectors, etc.
- Not obvious how to associate sensors and actuators – the pairing problem
- Computer control and the state feedback paradigm eventually led to centralized MIMO control

## Interaction between simple loops




$$Y_1(s) = P_{11}(s)U_1(s) + P_{12}U_2(s)$$
$$Y_2(s) = P_{21}(s)U_1(s) + P_{22}U_2(s),$$

What happens when the controllers are tuned individually?

## Rosenbrock's example

$$P(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Very benign subsystems (compare with example in [G&L, Ch.1]).

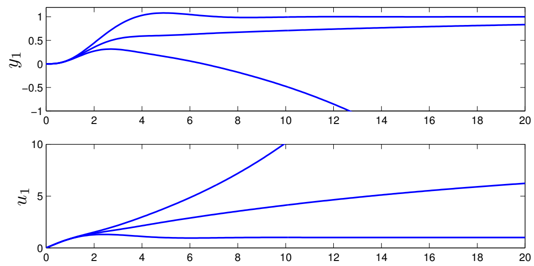
The transmission zeros are given by the roots of

$$\det P(s) = \frac{1}{s+1}\left(\frac{1}{s+1} - \frac{2}{s+3}\right) = \frac{1-s}{(s+1)^2(s+3)}$$

Difficult to control the system with a crossover frequency larger than $\omega_c \approx 0.5$.

## Rosenbrock's example with two SISO controllers

Controller $C_1$ is a PI controller with gains $k_1 = 1$, $k_i = 1$, and $C_2$ is a P controller with gains $k_2 = 0$, 0.8, or 1.6.




The second controller has a major impact on the first loop!

## Bristol's relative gain array (RGA)

- A simple way of measuring interaction based on static properties
- Edgar H. Bristol, "On a new measure of interaction for multivariable process control", [IEEE TAC 11(1967) pp. 133–135]
- Idea: What is effect of control of one loop on the steady state gain of another loop?

## RGA for $2 \times 2$ system

Consider the first loop $u_1 \to y_1$ when the second loop is in perfect control ($y_2 = 0$)

$$Y_1(s) = P_{11}(s)U_1(s) + P_{12}U_2(s)$$
$$0 = P_{21}(s)U_1(s) + P_{22}U_2(s).$$

Eliminating $U_2(s)$ from the first equation gives

$$Y_1(s) = \frac{P_{11}(s)P_{22}(s) - P_{12}(s)P_{21}(s)}{P_{22}(s)} U_1(s).$$

The ratio of the static gains of loop 1 when the second loop is open and closed is

$$\lambda = \frac{P_{11}(0)P_{22}(0)}{P_{11}(0)P_{22}(0) - P_{12}(0)P_{21}(0)} = \frac{1}{1 - \frac{P_{12}(0)P_{21}(0)}{P_{11}(0)P_{22}(0)}}$$

## Interpretation of RGA for $2 \times 2$ systems

$\lambda = 1$: No interaction

$\lambda = 0$: Open-loop gain $u_1 \to y_1$ is zero. Avoid this.

$0 < \lambda < 1$: Closed loop gain $u_1 \to y_1$ is larger than open loop gain.

$\lambda > 1$: Closed loop gain $u_1 \to y_1$ is smaller than open loop gain. Interaction increases with increasing $\lambda$. Very difficult to control both loops independently if $\lambda$ is very large.

$\lambda < 0$: The closed loop gain $u_1 \to y_1$ has different sign than the open loop gain. Opening or closing the second loop has dramatic effects. The loops are counteracting each other. Such pairings should be avoided for decentralized control and the loops should be controlled jointly as a multivariable system.

## RGA for square systems

Let $P(s)$ be an $n \times n$ transfer matrix. The relative gain array of $P(0)$ is

$$\Lambda = P(0) \,.\!*\, \left(P^{-1}(0)\right)^T$$

The product $.*$ is "element-by-element product" (Schur or Hadamard product, same notation in Matlab). Properties:

- $P$ diagonal or triangular gives $\Lambda = I$
- Not affected by diagonal scalings

Insight and use

- Tells how the static gain in one loop is influenced by perfect control in all other loops
- Dimension free. Row and column sums are 1.
- Elements close to 1 are good candidates for input–output pairing
- Negative elements correspond to sign reversals due to feedback of other loops – avoid!

## RGA for general systems

The RGA can be computed for a general transfer matrix $G$ at some frequency $\omega$:

$$\mathrm{RGA}(G(i\omega)) = G(i\omega) \,.\!*\, \left(G^{\dagger}(i\omega)\right)^T$$

$\dagger$ denotes the pseudo-inverse (Matlab: `pinv`)

Often, $\omega = 0$ and $\omega = \omega_c$ are investigated

## Another interpretation of RGA

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = P \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = P^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

- $P_{kj}$ gives the map $u_j \to y_k$ when $u_i = 0$ for $i \neq j$
- $[P^{-1}]_{jk}$ gives the map $y_k \to u_j$ when $y_i = 0$ for $i \neq k$

If $[RGA(P)]_{k,j} = 1$, then only $y_k$ is needed to recover $u_j$. This means strong coupling and $u_j$ is a natural input for control of $y_k$.

## Pairing

When designing complex systems loop by loop we must decide what measurements should be used as inputs for each controller. This is called the **pairing** problem. The choice can be governed by physics but the relative gain array can also be used

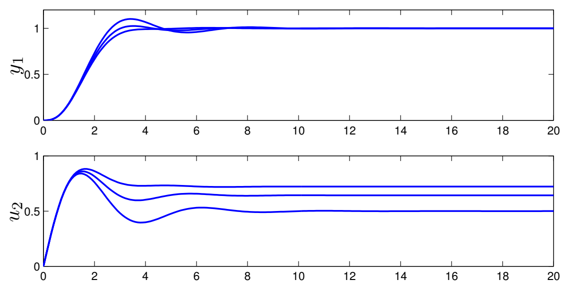
Consider Rosenbrock's example

$$P(0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad P^{-1}(0) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\Lambda = P(0) \,.\!*\, (P^{-1}(0))^T = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

- Negative sign indicates the sign reversal found previously
- Better to use reverse pairing, i.e. let $u_2$ control $y_1$

## Rosenbrock's example with reverse pairing




- $U_2 = \left(1 + \frac{1}{s}\right)(Y_{\text{ref}1} - Y_1)$
- $u_1 = -k_2 y_2$ with $k_2 = 0$, 0.8, and 1.6.

## Interactions can be beneficial

$$P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} = \begin{pmatrix} \dfrac{s-1}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \\ \dfrac{-6}{(s+1)(s+2)} & \dfrac{s-2}{(s+1)(s+2)} \end{pmatrix}.$$

RGA:

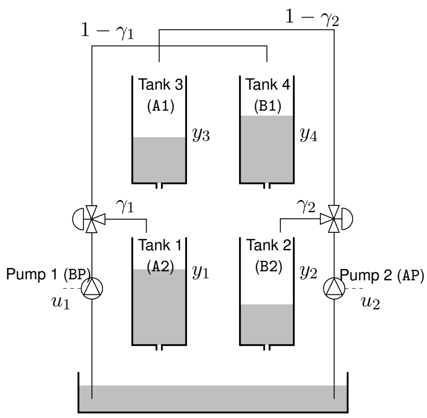
$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Transmission zeros

$$\det P(s) = \frac{(s-1)(s-2)+6s}{(s+1)^2(s+2)^2} = \frac{s^2+4s+2}{(s+1)^2(s+2)^2}$$

Difficult to control individual loops fast because of the zero at $s = 1$. Since there are no multivariable zeros in the RHP the multivariable system can easily be controlled fast (but this system is not robust to loop breaks)

## Lab 2: The quadruple tank

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## Decoupling

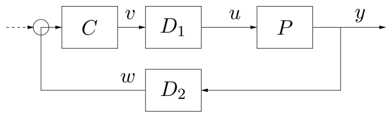
Simple idea: Find a compensator so that the system appears to be without coupling ("block-diagonal transfer function matrix").

Many versions:

- Input decoupling: $Q = PD_1$
- Output decoupling: $Q = D_2P$
- Both: $Q = D_2PD_1$

$D_1$ and $D_2$ can be static or dynamic systems

## Decoupling




Find $D_1$ and $D_2$ so that the controller sees a diagonal plant:

$$D_2PD_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a decentralized controller $C$ with same block-diagonal structure.

## Example: Input decoupling of $2 \times 2$ system

$$D_1(s) = \begin{bmatrix} 1 & -\frac{P_{12}}{P_{11}} \\ -\frac{P_{21}}{P_{22}} & 1 \end{bmatrix}$$