FRTN10 Multivariable Control, Lecture 8

Automatic Control LTH, 2016

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

6. Controllability, observability, multivariable zeros
7. Fundamental limitations
8. **Multivariable and decentralized control**

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Lecture 8 – Outline

1. Transfer functions for MIMO systems
2. Limitations due to RHP zeros
3. Decentralized control
4. Decoupling

See “Lecture notes” and [G&L, Chapters 1, 7.7 (first part) and 8.3]

Lecture 8 – Outline

1. Transfer functions for MIMO systems
2. Limitations due to RHP zeros
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Typical process control system

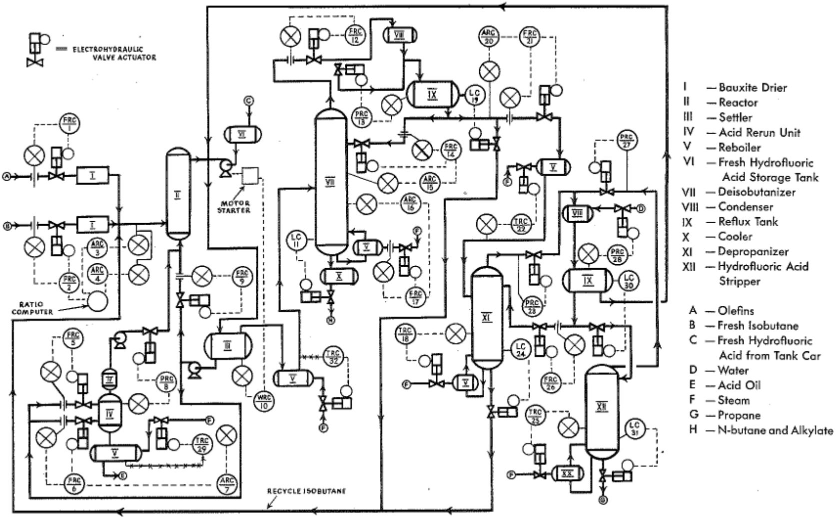
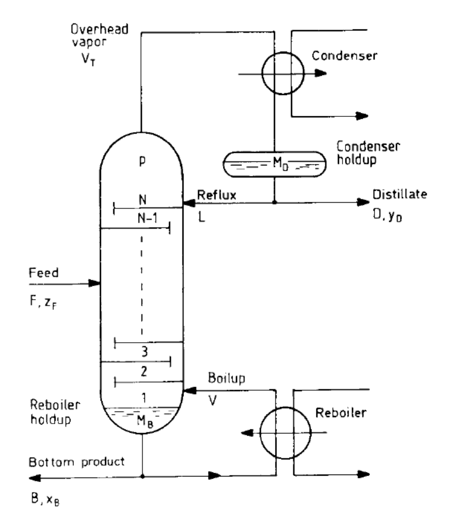


Figure 13–6. Automatic control system for Perco motor fuel alkylation process.

Example system: Distillation column



Raw oil inserted at bottom → different petro-chemical subcomponents extracted

Example system: Distillation column

Linear model:

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{4}{50s+1}e^{-27s} & \frac{1.8}{60s+1}e^{-28s} & \frac{5.9}{50s+1}e^{-27s} \\ \frac{5.4}{50s+1}e^{-18s} & \frac{5.7}{60s+1}e^{-14s} & \frac{6.9}{40s+1}e^{-15s} \end{bmatrix}}_{P(s)} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$$

Outputs:

y_1 = top draw composition

y_2 = side draw composition

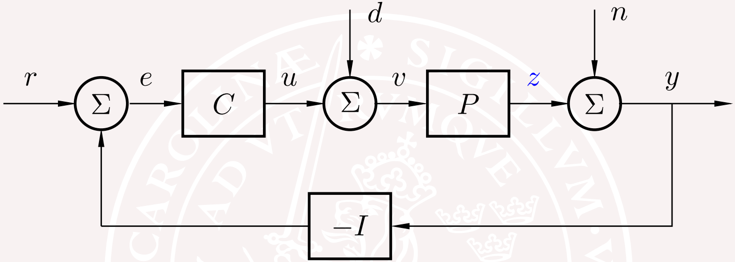
Inputs:

u_1 = top draw flowrate

u_2 = side draw flowrate

u_3 = bottom temperature control input

Multivariable transfer functions



P and C are matrices – order matters!

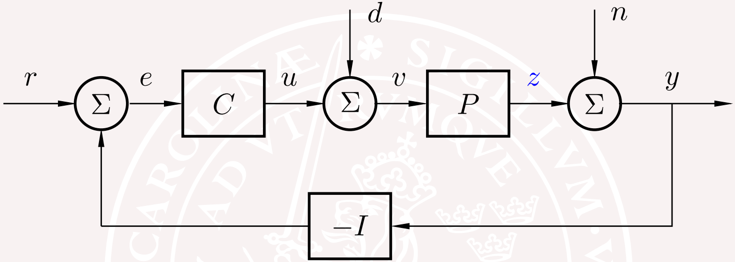
$$Z(s) = PC \cdot R(s) + P \cdot D(s) - PC \cdot [N(s) + Z(s)]$$

$$[I + PC]Z(s) = PC \cdot R(s) + P \cdot D(s) - PC \cdot N(s)$$

$$Z(s) = [I + PC]^{-1} \cdot (PC \cdot R(s) + P \cdot D(s) - PC \cdot N(s))$$

Notice that $[I + PC]^{-1}$ is generally not the same as $[I + CP]^{-1}$.

Multivariable transfer functions



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Notice that $[I + PC]^{-1}$ is generally not the same as $[I + CP]^{-1}$.

Sensitivity functions for MIMO systems

Output sensitivity function

$$S = (I + PC)^{-1}$$

$G_{? \to ?}$

Input sensitivity function

$$(I + CP)^{-1}$$

$G_{? \to ?}$

Complementary sensitivity function

$$T = (I + PC)^{-1} PC$$

$G_{? \to ?}$

Mini-problem:

Find the transfer functions above in the block diagram on the previous slide.

Sensitivity functions for MIMO systems

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Complementary sensitivity function

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Mini-problem:

Find the transfer functions above in the block diagram on the previous slide.

Some useful math relations

Notice the following identities:

$$(i)\quad [I+PC]^{-1}P = P[I+CP]^{-1}$$

$$(ii)\quad C[I+PC]^{-1} = [I+CP]^{-1}C$$

$$(iii)\quad T = P[I+CP]^{-1}C = PC[I+PC]^{-1} = [I+PC]^{-1}PC$$

$$(iv)\quad S+T = I$$

Proof:
The first equality follows by multiplication on both sides with $(I+PC)$ from the left and with $(I+CP)$ from the right.

Left: $[I+PC][I+PC]^{-1}P[I+CP] = I\cdot[P+PCP] = [I+PC]P$

Right: $[I+PC]P[I+CP]^{-1}[I+CP] = [I+PC]P\cdot I = [I+PC]P$

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Lecture 8 – Outline

Hard limitations from RHP zeros

[G&L Theorem 7.9]

Assume that the MIMO system $P(s)$ has a transfer zero z_i in the RHP.

Let $S(s) = [I + P(s)C(s)]^{-1}$ be the sensitivity function and let $W_S(s)$ be a scalar, stable and minimum phase transfer function. Then the specification

$$\|W_S S\|_\infty = \sup_\omega \bar{\sigma}\big(W_S(i\omega)S(i\omega)\big) \leq 1$$

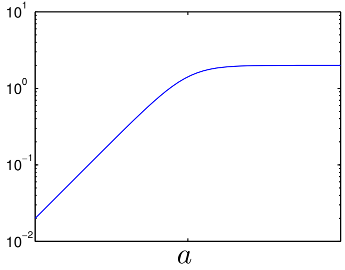
cannot be fulfilled unless

$$|W_S(z_i)| \leq 1$$

Example

Assume the specification $W_S(s) = \frac{s+a}{2s}$

$$|W_S(z_i)| = \frac{z_i + a}{2z_i} \le 1 \quad \Rightarrow \quad a \le z_i$$



Non-minimum-phase MIMO system

Example [G&L, Ch 1]

Consider a feedback system $Y(s) = (I + PC)^{-1}PCR(s)$ with the multivariable process

$$P(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Computing the determinant

$$\det P(s) = \frac{2}{(s+1)^2} - \frac{3}{(s+2)(s+1)} = \frac{-s+1}{(s+1)^2(s+2)}$$

shows that the process has a RHP zero at $s = 1$, which will limit the achievable performance.

See lecture notes for details of the following slides (checking three different controllers)

Non-minimum-phase MIMO system

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See lecture notes for details of the following slides (checking three different controllers)

Example – Controller 1

The controller

$$C_1(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & -\frac{3K_2(s+0.5)}{s(s+2)} \\ -\frac{K_1(s+1)}{s} & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

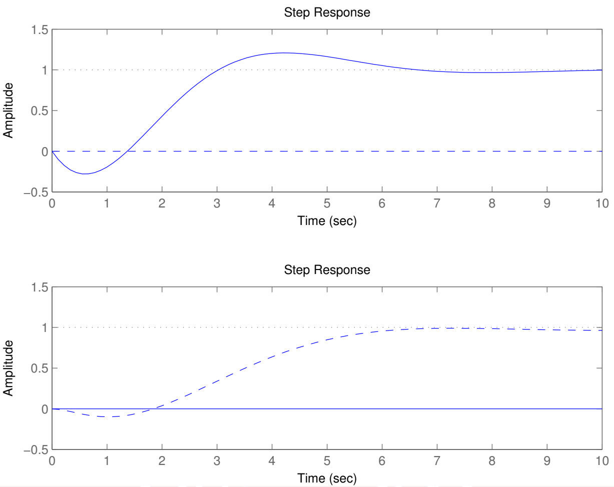
gives the diagonal loop transfer matrix

$$P(s)C_1(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & 0 \\ 0 & \frac{K_2(s+0.5)(-s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

Hence the system is decoupled into to scalar loops, each with an unstable zero at $s = 1$ that limits the bandwidth.

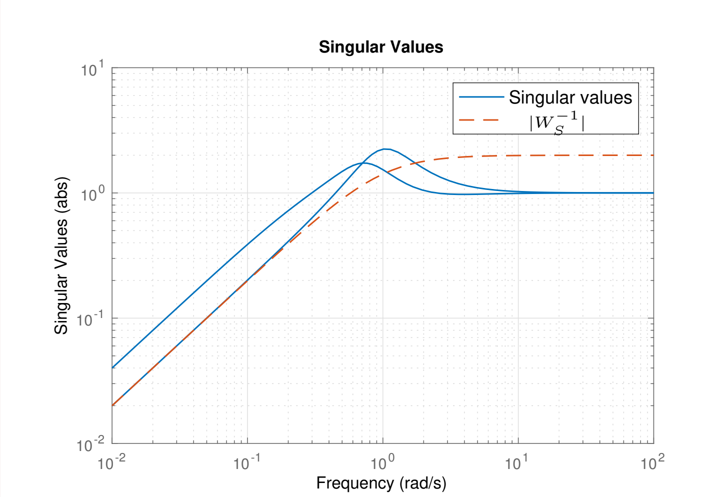
The closed-loop step responses for $K_1 = K_2 = 1$ are shown on next slide.

Step responses using Controller 1



Closed-loop step responses with decoupling controller $C_1(s)$ for the two outputs y_1 (solid) and y_2 (dashed). The upper plot is for a reference step for y_1. The lower plot is for a reference step for y_2.

Sensitivity sigma plot using Controller 1



$W_S(s) = \frac{s+1.001}{2s}$, impossible to meet

Example – Controller 2

The controller

$$C_2(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & K_2 \\ -\frac{K_1(s+1)}{s} & K_2 \end{bmatrix}$$

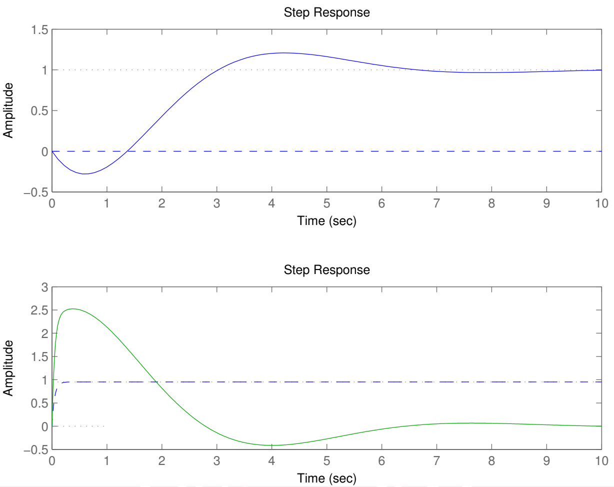
gives the triangular loop transfer matrix

$$P(s)C_2(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & \frac{K_2(5s+7)}{(s+2)(s+1)} \\ 0 & \frac{2K_2}{s+1} \end{bmatrix}$$

Now the decoupling is only partial:
Output y_2 is not affected by r_1. Moreover, there is no unstable zero that limits the rate of response in y_2!

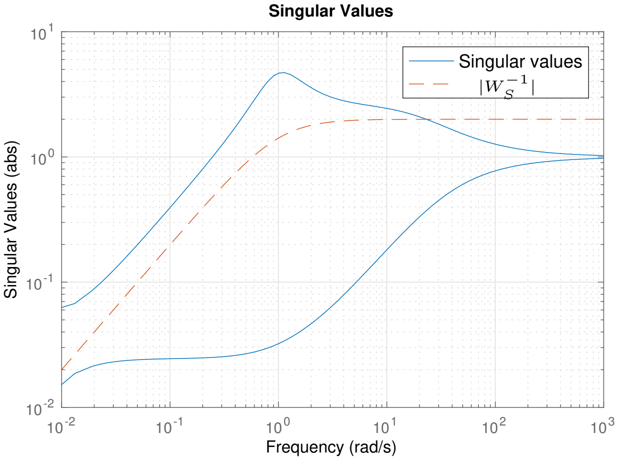
The closed loop step responses for $K_1 = 1$, $K_2 = 10$ are shown on next slide.

Step responses using Controller 2



Closed-loop step responses with controller $C_2(s)$ for the two outputs y_1 (solid) and y_2 (dashed). The RHP zero does not prevent a fast y_2-response to r_2 but at the price of a simultaneous undesired response in y_1.

Sensitivity sigma plot using Controller 2



$W_S(s) = \frac{s+1.001}{2s}$, impossible to meet

Example – Controller 3

The controller

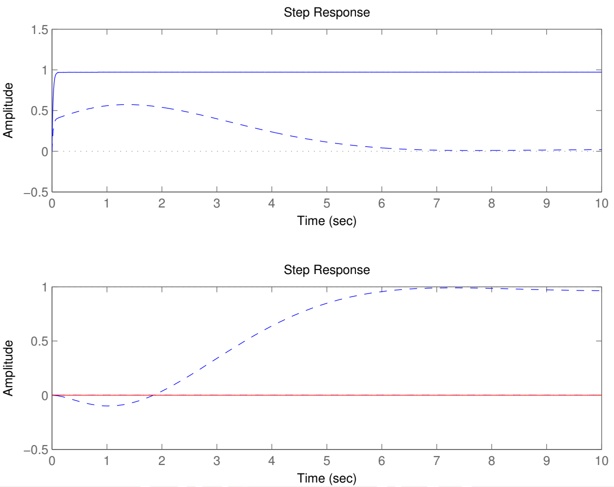
$$C_3(s) = \begin{bmatrix} K_1 & \frac{-3K_2(s+0.5)}{s(s+2)} \\ K_1 & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

gives the triangular loop transfer matrix

$$P(s)C_3(s) = \begin{bmatrix} \frac{K_1(5s+7)}{(s+1)(s+2)} & 0 \\ \frac{2K_1}{s+1} & \frac{K_2(-1+s)(s+0.5)}{s(s+1)^2(s+2)} \end{bmatrix}$$

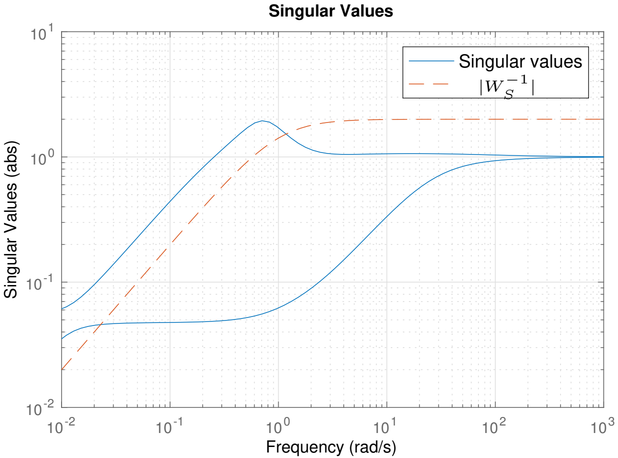
In this case y_1 is decoupled from r_2 and can respond arbitrarily fast for high values of K_1, at the expense of bad behavior in y_2. Step responses for $K_1 = 10$, $K_2 = 1$ are shown on next slide.

Step responses using Controller 3



Closed-loop step responses with controller $C_3(s)$ for the two outputs y_1 (solid) and y_2 (dashed). The RHP zero does not prevent a fast y_1-response to r_1 but at the price of a simultaneous undesired response in y_2.

Sensitivity sigma plot using Controller 3



$W_S(s) = \frac{s+1.001}{2s}$, impossible to meet

Example – summary

To summarize, the example shows that even though a **multivariable RHP zero always gives a performance limitation**, it is **possible to influence** where the effects should show up.

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1. Transfer functions for MIMO systems
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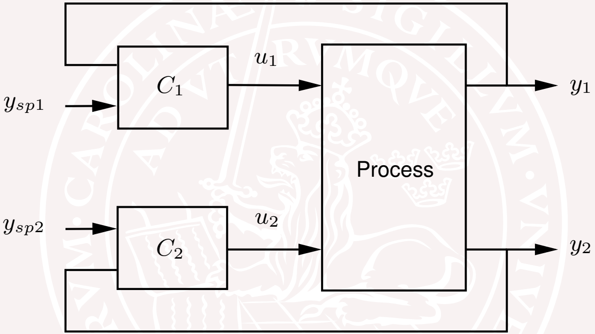
Decentralized control

Background in process control:

- A few important variables were controlled using the simple loop paradigm: one sensor, one actuator, one controller
- As more loops were added, interaction was handled using feedforward, cascade and midrange control, selectors, etc.
- Not obvious how to associate sensors and actuators – the pairing problem
- Computer control and the state feedback paradigm eventually led to centralized MIMO control

Interaction between simple loops



$$Y_1(s) = P_{11}(s)U_1(s) + P_{12}U_2(s)$$
$$Y_2(s) = P_{21}(s)U_1(s) + P_{22}U_2(s),$$

What happens when the controllers are tuned individually?

Rosenbrock's example

$$P(s) = \begin{pmatrix} \frac{1}{s+1} & \frac{2}{s+3} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Very benign subsystems (compare with example in [G&L, Ch.1]).

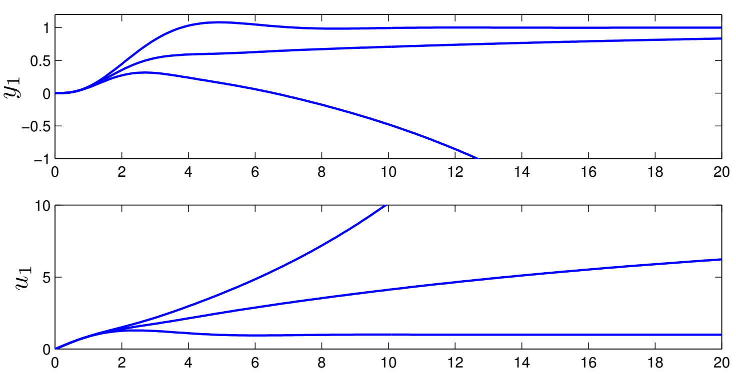
The transmission zeros are given by the roots of

$$\det P(s) = \frac{1}{s+1}\Big(\frac{1}{s+1} - \frac{2}{s+3}\Big) = \frac{1-s}{(s+1)^2(s+3)}$$

Difficult to control the system with a crossover frequency larger than $\omega_c \approx 0.5$.

Rosenbrock's example with two SISO controllers

Controller C_1 is a PI controller with gains $k_1 = 1$, $k_i = 1$, and C_2 is a P controller with gains $k_2 = 0$, 0.8, or 1.6.



The second controller has a major impact on the first loop!

Bristol's relative gain array (RGA)

- A simple way of measuring interaction based on static properties
- Edgar H. Bristol, "On a new measure of interaction for multivariable process control", [IEEE TAC 11(1967) pp. 133–135]
- Idea: What is effect of control of one loop on the steady state gain of another loop?

RGA for 2×2 system

Consider the first loop $u_1 \rightarrow y_1$ when the second loop is in perfect control ($y_2 = 0$)

$$\begin{aligned} Y_1(s) &= P_{11}(s)U_1(s) + P_{12}U_2(s) \\ 0 &= P_{21}(s)U_1(s) + P_{22}U_2(s). \end{aligned}$$

Eliminating $U_2(s)$ from the first equation gives

$$Y_1(s) = \frac{P_{11}(s)P_{22}(s) - P_{12}(s)P_{21}(s)}{P_{22}(s)} U_1(s).$$

The ratio of the static gains of loop 1 when the second loop is open and closed is

$$\lambda = \frac{P_{11}(0)P_{22}(0)}{P_{11}(0)P_{22}(0) - P_{12}(0)P_{21}(0)} = \frac{1}{1 - \frac{P_{12}(0)P_{21}(0)}{P_{11}(0)P_{22}(0)}}$$

Interpretation of RGA for 2×2 systems

$\lambda = 1$: No interaction

$\lambda = 0$: Open-loop gain $u_1 \to y_1$ is zero. Avoid this.

$0 < \lambda < 1$: Closed loop gain $u_1 \to y_1$ is larger than open loop gain.

$\lambda > 1$: Closed loop gain $u_1 \to y_1$ is smaller than open loop gain. Interaction increases with increasing λ. Very difficult to control both loops independently if λ is very large.

$\lambda < 0$: The closed loop gain $u_1 \to y_1$ has different sign than the open loop gain. Opening or closing the second loop has dramatic effects. The loops are counteracting each other. Such pairings should be avoided for decentralized control and the loops should be controlled jointly as a multivariable system.

RGA for square systems

Let $P(s)$ be an $n \times n$ transfer matrix. The relative gain array of $P(0)$ is

$$\Lambda = P(0) \mathbin{.*} \left(P^{-1}(0)\right)^T$$

The product $.*$ is "element-by-element product" (Schur or Hadamard product, same notation in Matlab). Properties:

- P diagonal or triangular gives $\Lambda = I$
- Not affected by diagonal scalings

Insight and use

- Tells how the static gain in one loop is influenced by perfect control in all other loops
- Dimension free. Row and column sums are 1.
- Elements close to 1 are good candidates for input–output pairing
- Negative elements correspond to sign reversals due to feedback of other loops – avoid!

RGA for general systems

The RGA can be computed for a general transfer matrix G at some frequency ω:

$$\mathrm{RGA}(G(i\omega)) = G(i\omega) \mathbin{.*} \left(G^{\dagger}(i\omega)\right)^{T}$$

$\dagger$ denotes the pseudo-inverse (Matlab: `pinv`)

Often, $\omega = 0$ and $\omega = \omega_c$ are investigated

Another interpretation of RGA

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = P \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = P^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

- P_{kj} gives the map $u_j \to y_k$ when $u_i = 0$ for $i \neq j$
- $[P^{-1}]_{jk}$ gives the map $y_k \to u_j$ when $y_i = 0$ for $i \neq k$

If $[RGA(P)]_{k,j} = 1$, then only y_k is needed to recover u_j. This means strong coupling and u_j is a natural input for control of y_k.

Pairing

When designing complex systems loop by loop we must decide what measurements should be used as inputs for each controller. This is called the **pairing** problem. The choice can be governed by physics but the relative gain array can also be used

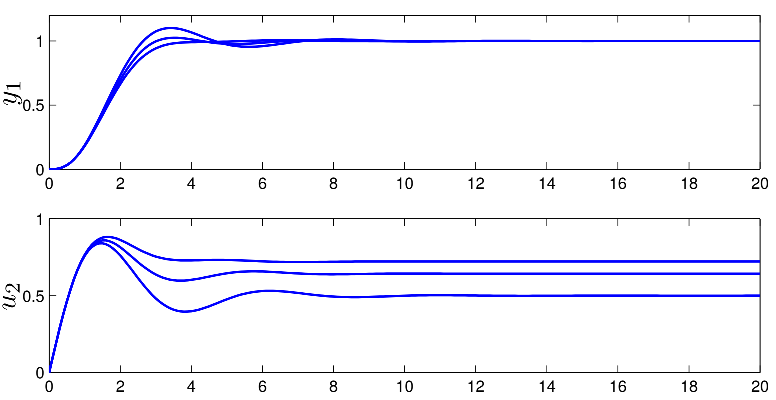
Consider Rosenbrock's example

$$P(0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad P^{-1}(0) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$\Lambda = P(0) \,.{*} (P^{-1}(0))^T = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

- Negative sign indicates the sign reversal found previously
- Better to use reverse pairing, i.e. let u_2 control y_1

Rosenbrock's example with reverse pairing



- $U_2 = \left(1 + \frac{1}{s}\right)(Y_{\text{ref1}} - Y_1)$
- $u_1 = -k_2 y_2$ with $k_2 = 0$, 0.8, and 1.6.

Interactions can be beneficial

$$P(s) = \begin{pmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{pmatrix} = \begin{bmatrix} \frac{s-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{-6}{(s+1)(s+2)} & \frac{s-2}{(s+1)(s+2)} \end{bmatrix}.$$

RGA:

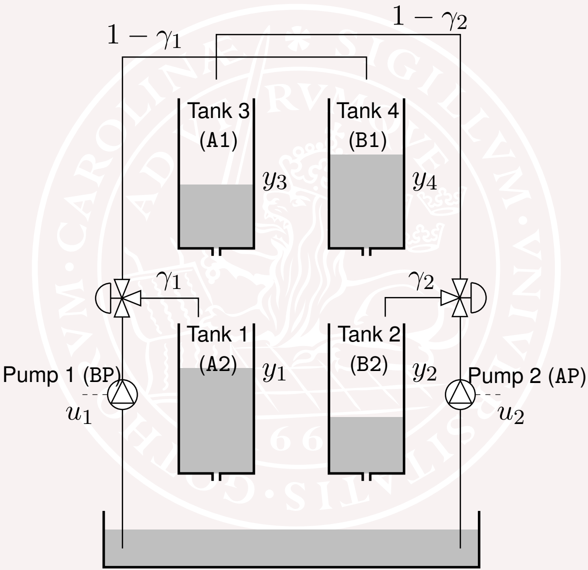
$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Transmission zeros

$$\det P(s) = \frac{(s-1)(s-2)+6s}{(s+1)^2(s+2)^2} = \frac{s^2+4s+2}{(s+1)^2(s+2)^2}$$

Difficult to control individual loops fast because of the zero at $s = 1$. Since there are no multivariable zeros in the RHP the multivariable system can easily be controlled fast (but this system is not robust to loop breaks)

Lab 2: The quadruple tank



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Decoupling

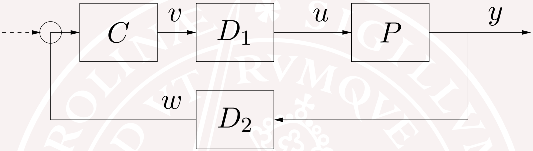
Simple idea: Find a compensator so that the system appears to be without coupling ("block-diagonal transfer function matrix").

Many versions:

- Input decoupling: $Q = PD_1$
- Output decoupling: $Q = D_2P$
- Both: $Q = D_2PD_1$

D_1 and D_2 can be static or dynamic systems

Decoupling



Find D_1 and D_2 so that the controller sees a diagonal plant:

$$D_2 P D_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a decentralized controller C with same block-diagonal structure.

Example: Input decoupling of 2×2 system

$$D_1(s) = \begin{bmatrix} 1 & -\frac{P_{12}}{P_{11}} \\ -\frac{P_{21}}{P_{22}} & 1 \end{bmatrix}$$