FRTN10 Multivariable Control, Lecture 7

Automatic Control LTH, 2016

Course Outline

- L1-L5 Specifications, models and loop-shaping by hand
- L6-L8 Limitations on achievable performance
  6. Controllability, observability, multivariable zeros
  7. **Fundamental limitations**
  8. Multivariable and decentralized control
- L9-L11 Controller optimization: Analytic approach
- L12-L14 Controller optimization: Numerical approach

Lecture 7 – Outline

1. Limitations from unstable poles and RHP zeros: Intuition
2. Example: A real-wheel-steered bicycle
3. Hard limitations from unstable poles/zeros
4. Bode's Relation and Bode's Integral Theorem

Limitations in control design

What we already know:

- Control signal limitations $\Rightarrow$ upper limit on bandwidth
- Model errors $\Rightarrow$ upper limit on bandwidth
- $S + T = 1$; S and T cannot both be small at the same frequency
- Some modes may be difficult (or even impossible) to control or observe due to lack of controllability or observability

Limitations in control design

Fundamental limitations:

- Unstable pole $\Rightarrow$ lower limit on bandwidth
- Right-half-plane (RHP) zero $\Rightarrow$ upper limit on bandwidth
- Time delay $\Rightarrow$ upper limit on bandwidth
- Amplitude and phase are coupled: Bode's relation
- S cannot be made small everywhere: Bode's integral theorem

Lecture 7 – Outline

Limitations from unstable poles and RHP zeros: Intuition

Example: A rear-wheel-steered bicycle

Hard limitations from unstable poles/zeros

Bode's Relation and Bode's Integral Theorem

Unstable poles – intuitive reasoning

An unstable pole p makes the output signal for a bounded input grow exponentially as $\sim e^{pt}$. To stabilize this system, one has to act fast, on a time scale $\sim 1/p$.

Intuitive conclusion: *Unstable poles give a lower bound on the speed of the closed loop.*

Systems with time-delay

Assume that the plant contains a time-delay T. This means e.g. that a load disturbance is not visible in the output signal until after at least T time units. Of course, this puts a hard constraint on how quickly a feedback controller can reject the disturbance!

Intuitive conclusion: *Time delays give an upper bound on the speed of the closed loop.*

RHP zeros – intuitive reasoning

The step response of a system with a process *zero in the right half plane* (RHP) (i.e, with positive real part) initially goes in the "wrong direction".

Intuitive conclusion: *RHP zeros give an upper bound on the speed of the closed loop.*

Why the wrong direction? Let z_i be a process zero in the RHP. If we look at the step response $y(t)$ and its Laplace transform $Y(s)$ we get

$$0 = Y(z_i) = \int_0^\infty y(t) \underbrace{e^{-z_i t}}_{>0} dt$$

Hence, $y(t)$ must take both positive and negative values!

Recall: Step response for non-minimum-phase system

In the basic course, how did we show that the step response went in the "wrong direction" for systems with one zero in the RHP?

Use the "Initial value theorem" (see Collection of Formulae)

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t)$$

and apply it to the output derivative $\dot{y}(t)$.

(That is, look at sign of $\dot{y}(0+)$ and compare it to sign of final value $\lim_{t\to\infty} y(t)$)

Mini-problems

1. Give examples of systems that initially respond in the "wrong" direction.
2. Which of the intuitive arguments can be applied to
   - an inverted pendulum?
   - a rear-wheel-steered bicycle?

Lecture 7 – Outline

Limitations from unstable poles and RHP zeros: Intuition

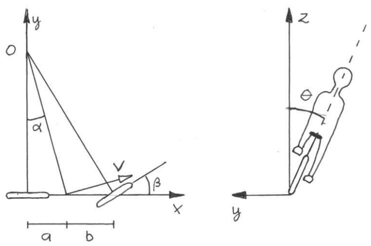
Example: A rear-wheel-steered bicycle

Hard limitations from unstable poles/zeros

Bode's Relation and Bode's Integral Theorem

Bike example

A (linearized) torque balance can be written as



$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

Bike example, cont'd

$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

where the physical parameters have typical values as follows:

Mass:	$m = 70$ kg
Distance rear-to-center:	$a = 0.3$ m
Height over ground:	$\ell = 1.2$ m
Distance center-to-front:	$b = 0.7$ m
Moment of inertia:	$J = 120$ kgm^2
Speed:	$V_0 = 5$ ms^{-1}
Acceleration of gravity:	$g = 9.81$ ms^{-2}

The transfer function from β to θ is

$$P(s) = \frac{mV_0\ell}{b}\frac{as + V_0}{Js^2 - mg\ell}$$

Bike example, cont'd

The system has an unstable pole p with time-constant

$$p^{-1} = \sqrt{\frac{J}{mg\ell}} \approx 0.4 \text{ s}$$

The closed loop system must be at least as fast as this. Moreover, the transfer function has a zero z with

$$z^{-1} = -\frac{a}{V_0} \approx -\frac{0.3 \text{ m}}{V_0}$$

For the back-wheel steered bike we have the same poles but different sign of V_0 and the zero will thus the be in the RHP!

An unstable pole-zero cancellation occurs for $V_0 \approx 0.75$m/s.

Lecture 7 – Outline

Limitations from unstable poles and RHP zeros: Intuition

Example: A rear-wheel-steered bicycle

Hard limitations from unstable poles/zeros

Bode's Relation and Bode's Integral Theorem

Sensitivity bounds from RHP zeros

It is easy to see that the sensitivity function must be equal to one at a RHP zero z_i of the transfer function:

$$P(z_i) = 0 \qquad \Rightarrow \qquad S(z_i) := \frac{1}{1 + \underbrace{P(z_i)}_{0} C(z_i)} = 1$$

Notice that the RHP zero in the plant can not be cancelled by an unstable pole in the controller, since this would give an unstable transfer function $C/(1 + PC)$ from measurement noise to control input.

Sensitivity bounds from unstable poles

Similarly, the complimentary sensitivity must be one at an unstable pole p_i:

$$P(p_i) = \infty \qquad \Rightarrow \qquad T(p_i) := \frac{P(p_i)C(p_i)}{1 + P(p_i)C(p_i)} = 1$$

In this case, cancellation by a RHP zero in the controller would give an unstable transfer function $P/(1 + PC)$ from input disturbance to plant output.

The Maximum Modulus Theorem

Suppose that all poles of the rational function $G(s)$ have negative real part. Then

$$\max_{\text{Re}\, s \geq 0} |G(s)| = \max_{\omega \in \mathbf{R}} |G(i\omega)|$$

Consequences of the Maximum Modulus Theorem

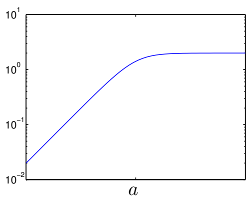
Assume that $W_S(s)$ and $W_T(s)$ are stable transfer functions. Then:

- The specification
  $$\|W_S S\|_\infty \leq 1$$
  cannot be met unless $|W_S(z_i)| \leq 1$ for all RHP zeros z_i
- The specification
  $$\|W_T T\|_\infty \leq 1$$
  cannot be met unless $|W_T(p_i)| \leq 1$ for all unstable poles p_i

Example: Hard limitation from RHP zero

Assume the sensitivity specification $W_S = \dfrac{s + a}{2s}$

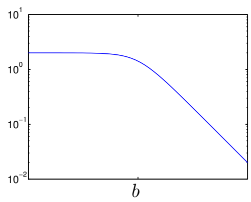
If the plant has a RHP zero z_i, then the specification is impossible to meet unless $a \leq z_i$



Example: Hard limitation from unstable pole

Assume the complementary sensitivity specification $W_T = \dfrac{s + b}{2b}$

If the plant has an unstable pole p_i, then the specification is impossible to meet unless $b \geq p_i$



Lecture 7 – Outline

Limitations from unstable poles and RHP zeros: Intuition

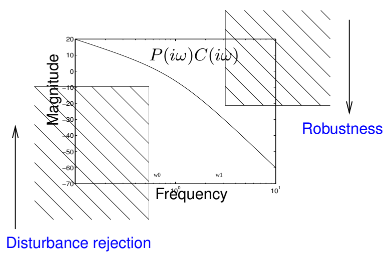
Example: A rear-wheel-steered bicycle

Hard limitations from unstable poles/zeros

Bode's Relation and Bode's Integral Theorem

Recall: Loop shaping design

The loop transfer function $L = PC$ should be made large at low frequencies and small at high frequencies:



How quickly can we make the transition from high to low gain and still retain a good phase margin?

Bode's Relation — approximate version

If $G(s)$ is proper, rational and stable with no RHP zeros, then

$$\arg G(i\omega_0) \approx \frac{\pi}{2} \left. \frac{d \log |G(i\omega)|}{d \log \omega} \right|_{\omega=\omega_0}$$

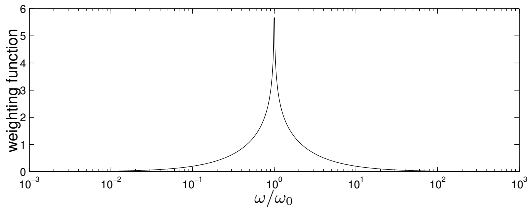
Otherwise the argument (phase) is even smaller.

As a consequence, the decay rate of the magnitude curve must be less than 2 at the cross-over frequency

Bode's Relation — exact version

If $G(s)$ is proper, rational and stable with no RHP zeros, then

$$\begin{aligned} \arg G(i\omega_0) &= \frac{2\omega_0}{\pi} \int_0^\infty \frac{\log |G(i\omega)| - \log |G(i\omega_0)|}{\omega^2 - \omega_0^2} d\omega \\ &= \frac{1}{\pi} \int_0^\infty \frac{d \log |G(i\omega)|}{d \log \omega} \underbrace{\log \left| \frac{\omega + \omega_0}{\omega - \omega_0} \right|}_{\text{weighting function}} d \log \omega \end{aligned}$$



Bode's Integral Theorem ("The water bed effect")

For a system with loop gain $L = PC$ which has a relative degree ≥ 2 and unstable poles $p_1, \ldots, p_M$, the following *conservation law* for the sensitivity function $S = \dfrac{1}{1+L}$ holds.

$$\int_0^{+\infty} \log |S(i\omega)| d\omega = \pi \sum_{i=1}^{M} \text{Re}(p_i)$$

See [G&L Theorem 7.3] for details.

(A similar condition relating T and RHP zeros exists, see [G&L Theorem 7.5])

G. Stein: "Conservation of 'dirt!'"

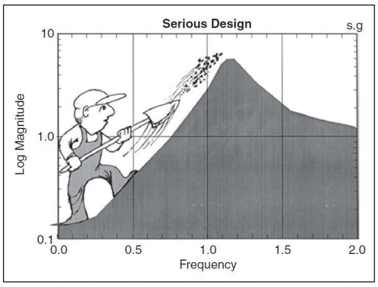


Figure 3. *Sensitivity reduction at low frequency unavoidably leads to sensitivity increase at higher frequencies.*

Picture from Gunter Stein's Bode Lecture (1985) "Respect the unstable". Reprint in *IEEE Control Systems Magazine*, Aug 2003.