FRTN10 Multivariable Control, Lecture 6

Automatic Control LTH, 2016

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

6. **Controllability/observability, multivariable poles/zeros, realizations**
7. Fundamental limitations
8. Multivariable and decentralized control

L9-L11 Controller optimization: Analytic approach

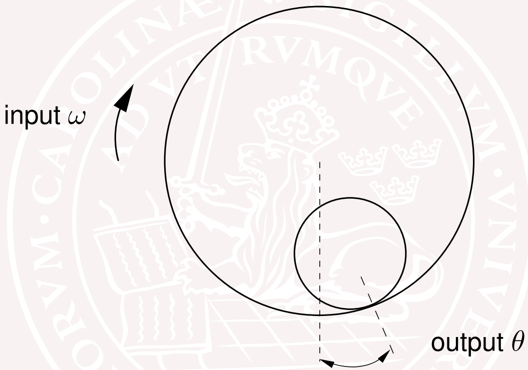
L12-L14 Controller optimization: Numerical approach

Lecture 6 – Outline

1. Controllability and observability, gramians
2. Multivariable poles and zeros
3. Minimal state-space realizations

[Glad & Ljung] Ch. 3.2–3.3, beg. of 3.5; Lecture notes on course web page

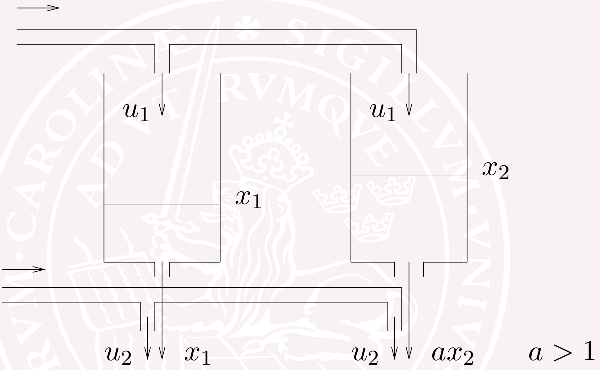
Example: Ball in the Hoop



$$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$$

Can you reach $\theta = \pi/4$, $\dot{\theta} = 0$? Can you stay there?

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$
$$\dot{x}_2 = -ax_2 + u_1 \qquad y_2 = ax_2 + u_2$$

Can you reach $y_1 = 1, y_2 = 2$? Can you stay there?

Lecture 6 – Outline

1. Controllability and observability
2. Multivariable poles and zeros
3. Minimal realizations

Remember: Controllability and state feedback

Process

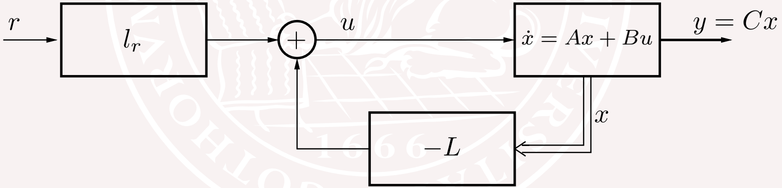
$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$$

Closed-loop system

$$\begin{cases} \dot{x} = (A - BL)x + Bl_r r \\ y = Cx \end{cases}$$

State-feedback control

$$u = -Lx + l_r r = -[l_1 \; l_2 \ldots l_n]x + l_r r$$



If the system (A, B) is *controllable* we can find a state feedback gain vector L to place the poles of the closed-loop system where we want

Remember: Observability and observers

Process

$$\begin{cases} \dfrac{dx}{dt} = Ax + Bu \\ y = Cx \end{cases}$$

Observer

$$\begin{cases} \dfrac{d\hat{x}}{dt} = \underbrace{A\hat{x} + Bu}_{\text{prediction}} + \underbrace{K(y - \hat{y})}_{\text{correction}} \\ \hat{y} = C\hat{x} \end{cases}$$

Estimation/observer error $\tilde{x} = x - \hat{x}$:

$$\begin{aligned} \dot{\tilde{x}} &= \dot{x} - \dot{\hat{x}} \\ &= Ax + Bu - A\hat{x} - Bu - K(Cx - C\hat{x}) \\ &= (A - KC)\tilde{x} \end{aligned}$$

If the system (A, C) is *observable* we can find an observer gain vector $K = \begin{bmatrix} k_1 \\ . \\ k_n \end{bmatrix}$ which assigns desired eigenvalues for $(A - KC)$.

Controllability

The system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

is **controllable** , if for every $x_1 \in \mathbf{R}^n$ there exists $u(t), t \in [0, t_1]$, such that $x(t_1) = x_1$ is reached from $x(0) = 0$.

The collection of vectors x_1 that can be reached in this way is called the **controllable subspace**.

(Matlab: `orth(ctrb(A,B))`)

Controllability criteria

The following statements regarding a system $\dot{x}(t) = Ax(t) + Bu(t)$ of order n are equivalent:

(i) The system is controllable

(ii) rank $[A - \lambda I \quad B] = n$ for all $\lambda \in \mathbf{C}$

(iii) rank $[B \;\; AB \ldots A^{n-1}B] = n$

If A is exponentially stable, define the **controllability Gramian**

$$S = \int_0^\infty e^{At} B B^T e^{A^T t} dt$$

For such systems there is a fourth equivalent statement:

(iv) The controllability Gramian is non-singular

Interpretation of the controllability Gramian

The controllability Gramian measures how difficult it is to reach a certain state.

In fact, let $S_1 = \int_0^{t_1} e^{At} B B^T e^{A^T t} dt$. Then, for the system $\dot{x}(t) = Ax(t) + Bu(t)$ to reach $x(t_1) = x_1$ from $x(0) = 0$ it is necessary that

$$\int_0^{t_1} |u(t)|^2 dt \geq x_1^T S_1^{-1} x_1$$

Equality is attained with

$$u(t) = B^T e^{A^T (t_1 - t)} S_1^{-1} x_1$$

(For proof, see the lecture notes.)

Computing the controllability Gramian

The controllability Gramian $S = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ can be computed by solving the Lyapunov equation

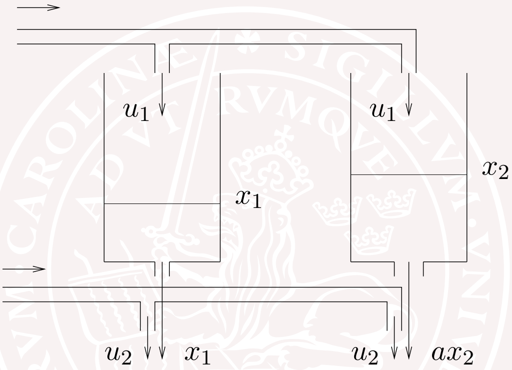
$$AS + SA^T + BB^T = 0$$

(For proof, see the lecture notes.)

Matlab: `S = lyap(A,B*B')`

Q: Where have we seen this equation before?

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

Controllability Gramian: $S = \int_0^\infty \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$

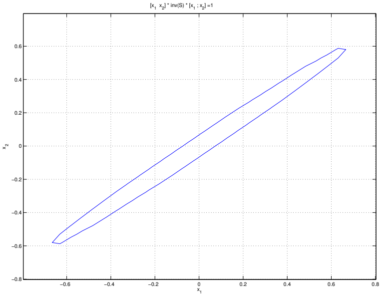
S close to singular when $a \approx 1$. Interpretation?

Example cont'd

Matlab:

```
>> a = 1.25;  A = [-1 0; 0 -1*a];  B=[1; 1];


>> Cs= [B  A*B], rank(Cs)
Cs =
    1.0000   -1.0000
    1.0000   -1.2500
ans =
     2
>> S = lyap(A,B*B')
S =
   0.5000    0.4444
   0.4444    0.4000
>> invS = inv(S)
invS =
    162.0   -180.0
   -180.0    202.5
```



Plot of $\begin{bmatrix} x_1 & x_2 \end{bmatrix} \cdot S^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1$

corresponds to the states we can reach by $\int_0^\infty |u(t)|^2 dt = 1.$

Observability

The system

$$\dot{x}(t) = Ax(t)$$
$$y(t) = Cx(t)$$

is **observable**, if the initial state $x(0) = x_0 \in \mathbf{R}^n$ can be uniquely determined by the output $y(t), t \in [0, t_1]$.

The collection of vectors x_0 that cannot be distinguished from $x = 0$ is called the **unobservable subspace**.

(Matlab: `null(obsv(A,C))`)

Observability criteria

The following statements regarding a system $\dot{x}(t) = Ax(t)$, $y(t) = Cx(t)$ of order n are equivalent:

(i) The system is observable

(ii) rank $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$ for all $\lambda \in \mathbf{C}$

(iii) rank $\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$

If A is exponentially stable, define the **observability Gramian**

$$O = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

For such systems there is a fourth equivalent statement:

(iv) The observability Gramian is non-singular

Interpretation of the observability Gramian

The observability Gramian measures how difficult it is to distinguish two initial states from each other by observing the output.

In fact, let $O_1 = \int_0^{t_1} e^{A^T t} C^T C e^{At} dt$. Then, for $\dot{x}(t) = Ax(t)$, the influence from the initial state $x(0) = x_0$ on the output $y(t) = Cx(t)$ satisfies

$$\int_0^{t_1} |y(t)|^2 dt = x_0^T O_1 x_0$$

Computing the observability Gramian

The observability Gramian $O = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$ can be computed by solving the Lyapunov equation

$$A^TO + OA + C^TC = 0$$

Matlab: `O = lyap(A',C'*C)`

Mini-problem

Is the water tank system with $a = 1$ observable?

What if only y_1 is available?

Lecture 6 – Outline

1. Controllability and observability
2. Multivariable poles and zeros
3. Minimal realizations

Poles and zeros

$$Y(s) = \underbrace{[C(sI - A)^{-1}B + D]}_{G(s)} U(s)$$

For scalar systems, the points $p \in \mathbf{C}$ where $G(s) = \infty$ are called **poles** G. They are eigenvalues of A and determine stability.

The poles of $G(s)^{-1}$ are called **zeros** of G.

This definition can be used also for square MIMO systems, but we will next give a more general definition, involving also multiplicity.

Pole and zero polynomials

- The **pole polynomial** is the least common denominator of all minors (sub-determinants) to $G(s)$.
- The **zero polynomial** is the greatest common divisor of the maximal minors of $G(s)$.

The **poles** of G are the roots of the pole polynomial.

The **(transmission) zeros** of G are the roots of the zero polynomial.

Poles and zeros – example

Calculate the poles and zeros of

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

Poles: Minors: $\frac{2}{s+1}, \frac{3}{s+2}, \frac{1}{s+1}, \frac{1}{s+1}, \frac{2}{(s+1)^2} - \frac{3}{(s+1)(s+2)} = \frac{-(s-1)}{(s+1)^2(s+2)}$

The least common denominator is $(s+1)^2(s+2)$, giving the poles -2 and -1 (double)

Zeros: Maximal minor: $\frac{-(s-1)}{(s+1)^2(s+2)}$

The greatest common divisor is $s-1$, giving the (transmission) zero 1.

Poles and zeros – example

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Zeros: Maximal minor: $\frac{-(s-1)}{(s+1)^2(s+2)}$

The greatest common divisor is $s-1$, giving the (transmission) zero 1.

Zeros of square systems

When $G(s)$ is square, the only maximal minor is $\det G(s)$, so the zeros are determined from the equation

$$\det G(s) = 0$$

For a square system with minimal state-space realization, the zeros are the solutions to

$$\det \begin{bmatrix} sI - A & B \\ -C & D \end{bmatrix} = 0$$

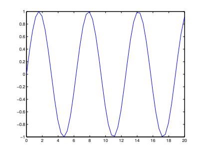
Interpretation of poles and zeros

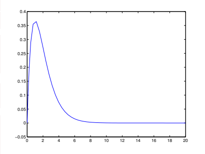
Poles:

- A pole $s = a$ is associated with a time function $x(t) = x_0 e^{at}$
- A pole $s = a$ is an eigenvalue of A

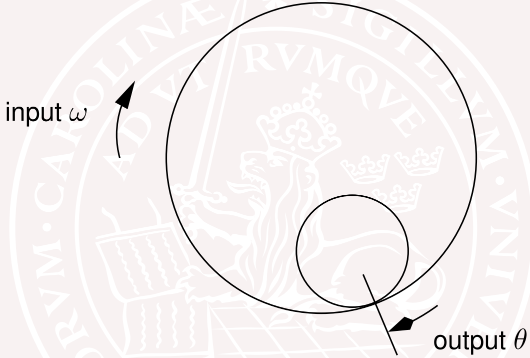
Zeros:

- A zero $s = a$ means that an input $u(t) = u_0 e^{at}$ is blocked
- A zero describes how inputs and outputs couple to states





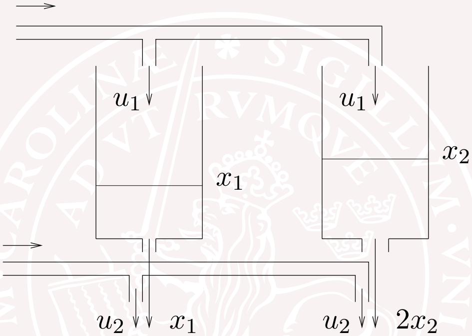
Example: Ball in the Hoop



$$\ddot{\theta} + c\dot{\theta} + k\theta = \dot{\omega}$$

The transfer function from ω to θ is $\frac{s}{s^2+cs+k}$. The zero in $s = 0$ makes it impossible to control the stationary position of the ball.

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad y_1 = x_1 + u_2$$

$$\dot{x}_2 = -2x_2 + u_1 \qquad y_2 = 2x_2 + u_2$$

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix} \qquad \det G(s) = \frac{-s}{(s+1)(s+2)}$$

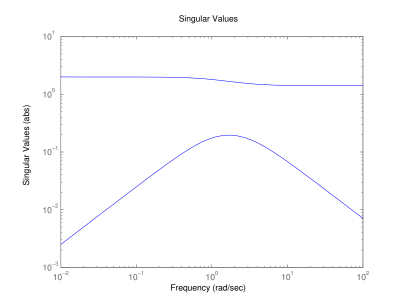
The system has a zero in the origin! At stationarity $y_1 = y_2$.

Plot singular values of $G(i\omega)$ vs frequency

```
» s=tf('s')
» G=[1/(s+1) 1 ; 2/(s+2) 1]
» sigma(G) ; plot singular values

% Alt. for a certain frequency:

» w=1;
» A = freqresp(G,i*w);
» [U,S,V] = svd(A)
```



The largest singular value of $G(i\omega) = \begin{bmatrix} \frac{1}{i\omega+1} & 1 \\ \frac{2}{i\omega+2} & 1 \end{bmatrix}$ is fairly constant.

This is due to the second input. The first input makes it possible to control the difference between the two tanks, but mainly near $\omega = 1$ where the dynamics make a difference.

Lecture 6 – Outline

1. Controllability and observability
2. Multivariable poles and zeros
3. Minimal realizations

Realization in diagonal form

Consider a transfer matrix with partial fraction expansion

$$G(s) = \sum_{i=1}^{n} \frac{C_i B_i}{s - p_i} + D$$

This has the realization

$$\dot{x}(t) = \begin{bmatrix} p_1 I & & 0 \\ & \ddots & \\ 0 & & p_n I \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ \vdots \\ B_n \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} C_1 & \dots & C_n \end{bmatrix} x(t) + Du(t)$$

The rank of the matrix $C_i B_i$ determines the necessary number of columns in B_i and the multiplicity of the pole p_i.

(Warning: Matlab has no good command for doing this)

Realization of multivariable system – example 1

To find a minimal state-space realization for the system

$$G(s) = \begin{pmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{pmatrix}$$

with poles in -2 and -1 (double), write the transfer matrix as (e.g.)

$$G(s) = \frac{\begin{bmatrix} 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 3 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix}}{s+2}$$

giving the realization

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 2 & 0 & 3 \\ 1 & 1 & 0 \end{pmatrix} x$$

Realization of multivariable system – example 2

To find state space-realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+3)} \\ \frac{6}{(s+2)(s+4)} & \frac{1}{s+2} \end{bmatrix}$$

write the transfer matrix as

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+3} \\ \frac{3}{s+2} - \frac{3}{s+4} & \frac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix}}{s+1} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}3 & 1\end{bmatrix}}{s+2} + \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0 & -1\end{bmatrix}}{s+3} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}-3 & 0\end{bmatrix}}{s+4}$$

This gives the realization

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$