## FRTN10 Multivariable Control, Lecture 4

**Automatic Control LTH, 2016**

## Course Outline

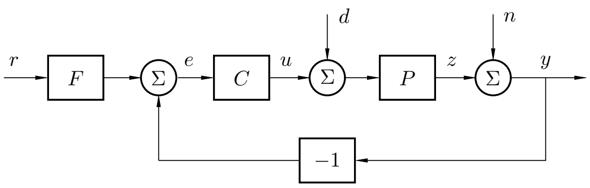


## Lecture 4 – Outline

1. Frequency domain specifications
2. Loop shaping
3. Feedforward design

[Glad & Ljung] Ch. 6.4–6.6, 8.1–8.2

## Relations between signals




$$Z = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$
$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$
$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

## Design specifications

Find a controller that

**A:** reduces the effect of load disturbances

**B:** does not inject too much measurement noise into the system

**C:** makes the closed loop insensitive to process variations

**D:** makes the output follow the setpoint

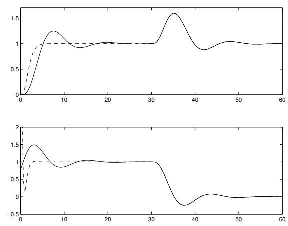
If possible, use a controller with **two degrees of freedom**, i.e. separate signal transmission from $y$ to $u$ and from $r$ to $u$. This gives a nice separation of the design problem:

1. Design feedback to deal with A, B, and C
2. Design feedforward to deal with D

## Time domain specifications

E.g. specifications on step responses (w.r.t. reference, load disturbance)

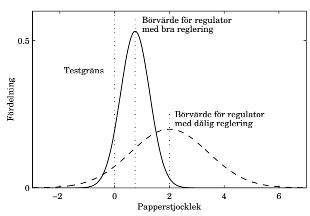
- Rise-time $T_r$
- Overshoot $M$
- Settling time $T_s$
- Static error $e_0$
- ...



reference change     load disturbance

## Stochastic signal specifications

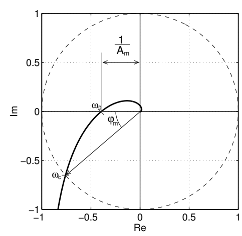
- Output variance
- Control signal variance
- ...




## Frequency domain specifications

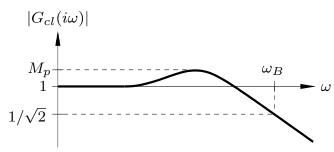
Open-loop specifications

- Amplitude margin $A_m$, phase margin $\varphi_m$
- Cross-over frequency $\omega_c$
- $M_s$ and $M_t$ circles in Nyquist diagram
- ...



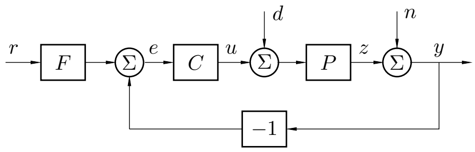

Closed-loop specifications, e.g.

- resonance peak $M_p$
- bandwidth $\omega_B$
- ...



## Frequency domain specifications

Closed-loop specifications, cont'd:




Desired properties:

- Fast tracking of setpoint $r$
- Small influence of load disturbance $d$ on $z$
- Small influence of model errors on $z$
- Limited amplification of noise $n$ in control $u$
- Robust stability despite model errors

## Frequency domain specifications

Ideally, we would like to design the controller ($C$ and $F$) so that

- $\frac{PCF}{1+PC} = 1$
- $\underbrace{\frac{P}{1+PC}}_{=PS} = \underbrace{\frac{1}{1+PC}}_{=S} = \underbrace{\frac{C}{1+PC}}_{=P^{-1}T} = \underbrace{\frac{PC}{1+PC}}_{=T} = 0$

$S + T = 1$ and other constraints makes this is impossible to achieve.

Typical compromise:

- Make $T$ small at high frequencies ($\omega > \omega_B$)
- Make $S$ small at low frequencies (+ possibly other disturbance dominated frequencies)

## Expressing specifications on $S$ and $T$

Maximum sensitivity specifications, e.g.,

- $\|S\|_\infty \le M_s$
- $\|T\|_\infty \le M_t$

Frequency-weighted specifications, e.g.,

- $\|W_S S\|_\infty \le 1$ or $|S(i\omega)| \le |W_S^{-1}(i\omega)|, \ \forall \omega$
- $\|W_T T\|_\infty \le 1$ or $|T(i\omega)| \le |W_T^{-1}(i\omega)|, \ \forall \omega$

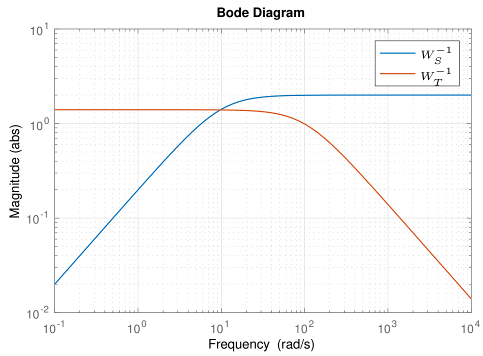
where $W_S(s)$ and $W_T(s)$ are stable transfer functions

Piecewise specifications, e.g.

- $|S(i\omega)| < \frac{0.2}{\omega}, \ \omega \le 10$ and $|S(i\omega)| < 2, \ \omega > 10$

## Specifications on $S$ and $T$ – example

$$W_S^{-1}(s) = \frac{2s}{s+10}, \quad W_T^{-1}(s) = \frac{140}{s+100}$$




## Limitations on specifications

The specifications cannot be chosen independently of each other:

- $S + T = 1$

Fundamental limitations [Lecture 7]:

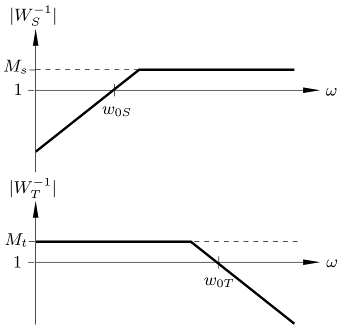
- RHP zero at $z \Rightarrow \omega_{0S} \le z/2$
- Time delay $T \Rightarrow \omega_{0S} \le 1/T$
- RHP pole at $p \Rightarrow \omega_{0T} \ge 2p$

Bode's integral theorem:

- The "waterbed effect"

Bode's relation:

- good phase margin requires certain distance between $\omega_{0S}$ and $\omega_{0T}$




## Loop shaping

Idea: Look at the **loop gain** $L = PC$ for design and to translate specifications on $S$ and $T$ into specifications on $L$

$$S = \frac{1}{1+L} \approx \frac{1}{L} \quad \text{if } L \text{ is large}$$

$$T = \frac{L}{1+L} \approx L \quad \text{if } L \text{ is small}$$

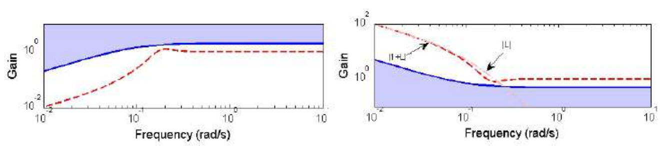
Classical loop shaping: Manually design $C$ so that $L = PC$ satisfies constraints on $S$ and $T$

- how are the specifications related?
- what to do with the region around cross-over frequency $\omega_c$ (where $|L| \approx 1$)?

## Sensitivity vs loop gain

$$S = \frac{1}{1+L}$$

$$|S(i\omega)| \le |W_S^{-1}(i\omega)| \iff |1 + L(i\omega)| > |W_S(i\omega)|$$



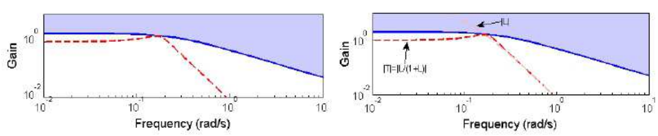

For small frequencies, $W_S$ large $\Longrightarrow 1 + L$ large, and $|L| \approx |1 + L|$.

$$|L(i\omega)| \ge |W_S(i\omega)| \quad (approx.)$$

## Complementary sensitivity vs loop gain

$$T = \frac{L}{1+L}$$

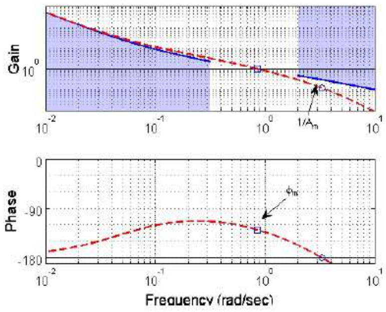
$$|T(i\omega)| \le |W_T^{-1}(i\omega)| \iff \frac{|L(i\omega)|}{|1 + L(i\omega)|} \le |W_T^{-1}(i\omega)|$$




For large frequencies, $W_T^{-1}$ small $\Longrightarrow |T| \approx |L|$

$$|L(i\omega)| \le |W_T^{-1}(i\omega)| \quad (approx.)$$

Resulting constraints on loop gain $L$:

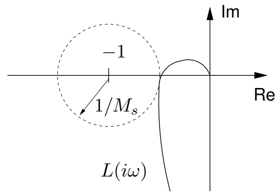


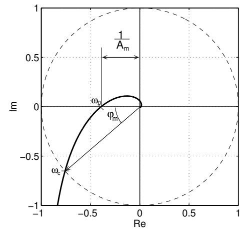

Approximations are inexact around cross-over frequency $\omega_c$. In this region, focus is on stability margins ($A_m$, $\varphi_m$)

## $M_s$ and $M_t$ vs gain and phase margins

Specifying $|S(i\omega)| \leq M_s$ and $|T(i\omega)| \leq M_t$ gives bounds for the gain and phase margins (but not the other way round!)

$$|S(i\omega)| \leq M_s \quad \Longrightarrow \quad A_m > \frac{M_s}{M_s - 1}, \quad \varphi_m > 2 \arcsin \frac{1}{M_s}$$




(Q: Why do not $A_m$ and $\varphi_m$ give bounds on $M_s$ and $M_t$?)

## Lead–lag compensation

Shape the loop gain $L = PC$ using a compensator $C$ composed of

- Lag (phase retarding) elements

$$C_{lag}(s) = \frac{s+a}{s+a/M}, \quad M > 1$$

- Lead (phase advancing) elements

$$C_{lead}(s) = N\frac{s+b}{s+bN}, \quad N > 1$$

- Gain

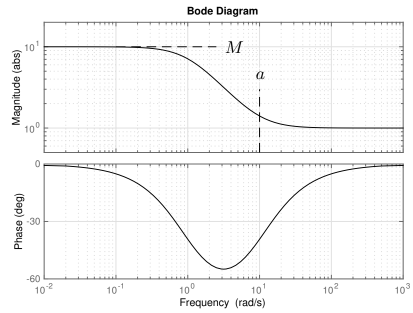
$$K$$

Example:

$$C(s) = K\frac{s+a}{s+a/M} \cdot N\frac{s+b}{s+bN}$$

## Lag filter

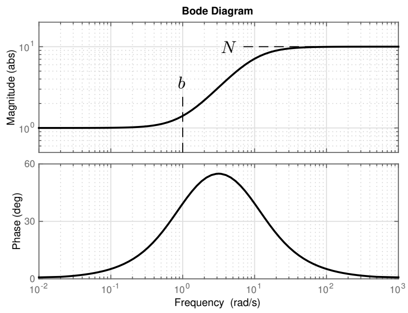
$$G_{lag}(s) = \frac{s+a}{s+a/M}, \quad M > 1$$




Special case: $M = \infty \Rightarrow$ integrator

## Lead filter

$$G_{lead}(s) = N\frac{s+b}{s+bN}, \quad N > 1$$




Maximum phase advance for different $N$ given in Collection of Formulae

## Properties of lead–lag filters

- Lag element
  - Reduces static error
  - Reduces stability margin
- Lead element
  - Increases speed (by increasing $\omega_c$)
  - Increased phase
    $\Longrightarrow$ May improve stability
- Gain
  - Translates the magnitude curve
  - Does not change phase curve

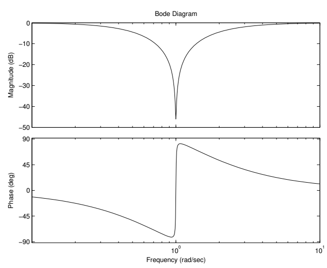
## Iterative lead–lag design

- Step 1: Lag (phase retarding) element
  - Add phase retarding element to get low-frequency asymptote right
- Step 2: Phase advancing element
  - Use phase advancing element to obtain correct phase margin
- Step 3: Adjust gain
  - Usually need to amplitude curve to obtain the desired cross-over frequency.

Adjusting the gain in Step 3 leaves the phase unaffected, but may ruin low-frequency asymptote (need to revise lag element) $\Longrightarrow$ Need to iterate!

Example of other compensation link:

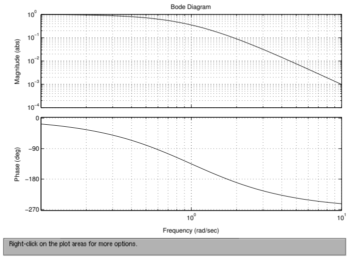
Notch filter $\dfrac{s^2 + 0.01s + 1}{s^2 + 2s + 1}$

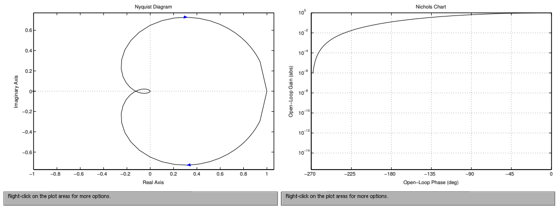



(E.g., supress measurement noise at specific frequency)

Bode, Nyquist and Nichols diagrams

$\log|PC| = \log|P| + \log|C|$
$\arg\{PC\} = \arg\{P\} + \arg\{C\}$

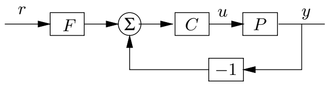


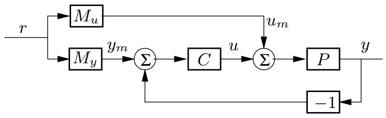

## Feedforward design

Examples of 2-DOF configurations:

(1)

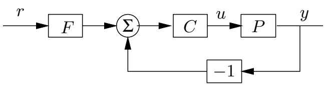



(2)




Ideally, we would like the output to follow the setpoint perfectly, i.e. $y = r$

## Feedforward design (1)



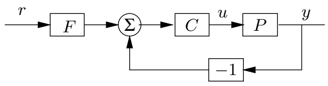

Perfect following requires

$$F = \frac{1 + PC}{PC} = T^{-1}$$

In general impossible because of pole excess in $T$. Also

- $T$ might contain non-minimum-phase factors that can/should not be inverted
- $u$ must typically satisfy some upper and lower limits

## Feedforward design (1)

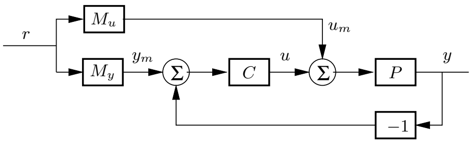



Assume $T$ minimum phase. An implementable choice of $F$ is then

$$F(s) = \frac{1 + P(s)C(s)}{P(s)C(s)(sT_f + 1)^d}$$

where $d$ is large enough to make $F$ proper and implementable

## Feedforward design (2)

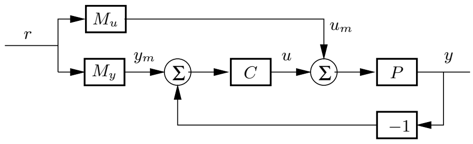



$M_y$ and $M_u$ can be viewed as generators of the desired output $y_m$ and the input $u_m$ that corresponds to $y_m$

For $y$ to follow $y_m$, select

$$M_u = M_y/P$$

## Feedforward design (2)




Since $M_u = M_y/P$ should be stable, causal and proper we find that

- Unstable zeros of $P$ must be zeros of $M_y$
- Time delays of $P$ must be time delays of $M_y$
- The pole excess of $M_y$ must not be smaller than the pole excess of $P$

Take process limitations into account!

## Feedforward design – example

Process:

$$P(s) = \frac{1}{(s+1)^4}$$

Selected reference model:

$$M_y(s) = \frac{1}{(sT+1)^4}$$

Then

$$M_u(s) = \frac{M_y(s)}{P(s)} = \frac{(s+1)^4}{(sT_f+1)^4} \qquad M_u(\infty) = \frac{1}{T^4}$$

Fast response (small $T_f$) requires high gain of $M_u$.

Bounds on the control signal limit how fast response we can obtain.