FRTN10 Multivariable Control, Lecture 3

Automatic Control LTH, 2016

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

1. Introduction
2. Stability and robustness
3. **Specifications and disturbance models**
4. Control synthesis in frequency domain
5. Case study

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Lecture 3 – Outline

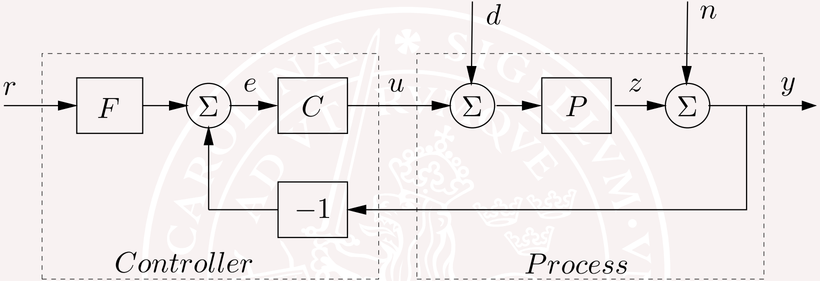
1. Specifications
2. Stochastic disturbances
3. Filtering of white noise

[Glad & Ljung] Ch. 5.1–5.6, 6.1–6.3

Lecture 3 – Outline

1. Specifications
2. Disturbance models
3. Filtering of white noise

A basic control system

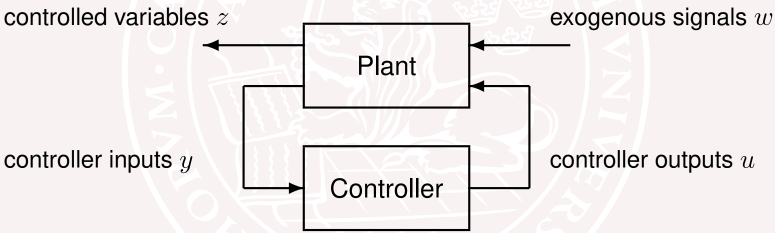


Ingredients:

- Controller: feedback C, feedforward F
- Load disturbance d: drives the system from desired state
- Process: transfer function P
- Process variable z should follow reference r
- Measurement noise n: corrupts information about z

A more general setting

Load disturbances need not enter at the process input, and measurement noise and setpoint values may also enter in different ways. More general setting:



We will return to this setting later in the course

Design specifications

Find a controller that

- **A:** reduces the effect of load disturbances
- **B:** does not inject too much measurement noise into the system
- **C:** makes the closed loop insensitive to process variations
- **D:** makes the output follow the setpoint

If possible, use a controller with **two degrees of freedom** (2 DOF), i.e. separate signal transmission from y to u and from r to u. This gives a nice separation of the design problem:

1. Design feedback to deal with A, B, and C
2. Design feedforward to deal with D

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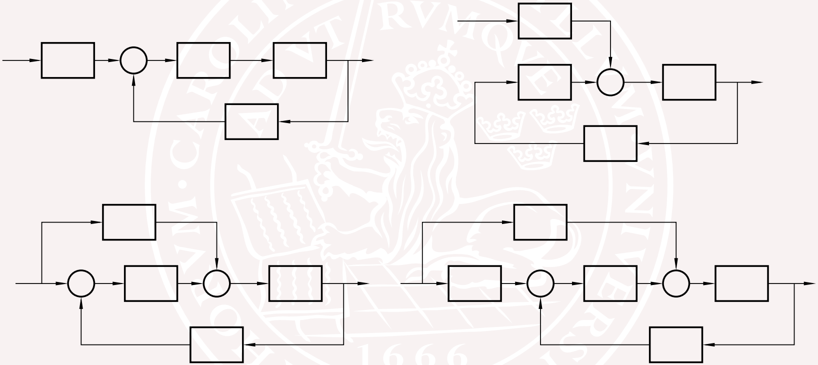
2-DOF Control Structures

A 2-DOF controller can be represented in many different ways, e.g.:

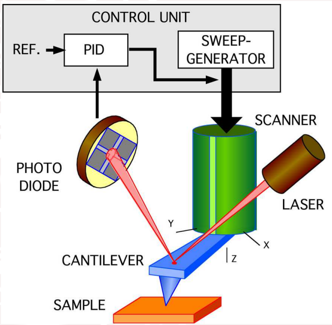


For linear systems, all of these structures are equivalent

Some systems only allow error feedback

Disk drive

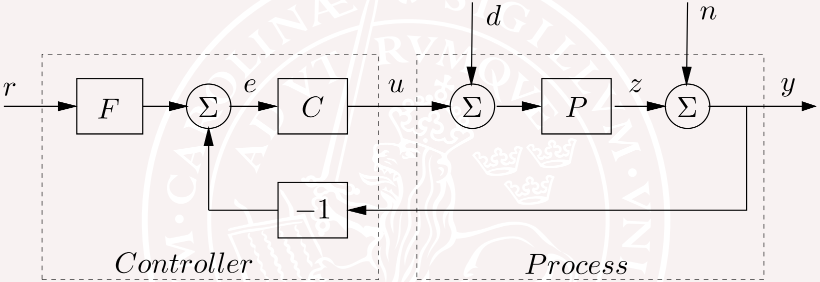
Atomic Force Microscope



Only the control error can be measured

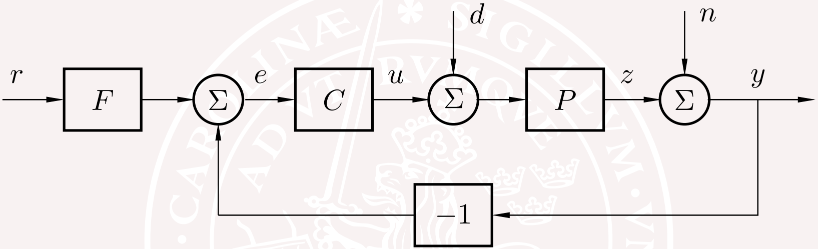
Design of disturbance attenuation and setpoint response cannot be separated

Mini-problem



What is the transfer function from measurement noise to control signal? From load disturbance to process output?

Relations between signals



$$Z = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

The Gang of Four / Gang of SIx

Four transfer functions are needed to characterize the response to load disturbances and measurement noise:

$$
\begin{array}{cc}
\dfrac{PC}{1+PC} & \dfrac{P}{1+PC} \\
\dfrac{C}{1+PC} & \dfrac{1}{1+PC}
\end{array}
$$

Two more are required to describe the response to setpoint changes (for 2-DOF controllers):

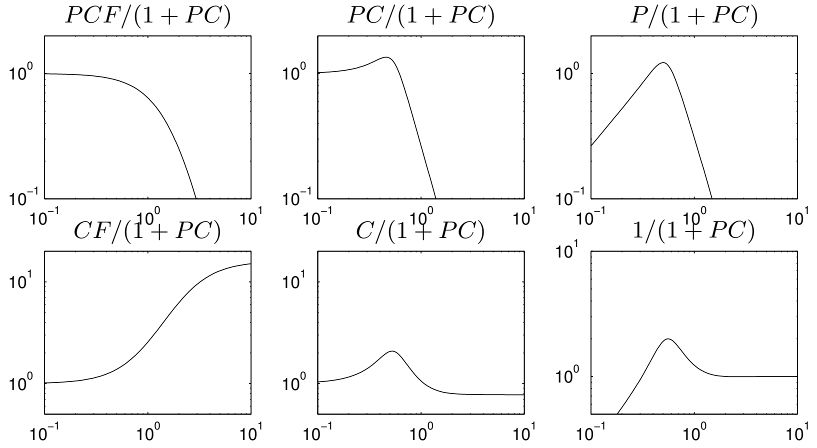
$$
\dfrac{PCF}{1+PC} \qquad \dfrac{CF}{1+PC}
$$

Some observations

- To fully understand a system it is necessary to look at **all** four/six transfer functions
- It may be strongly misleading to show properties of only one or a few transfer functions, for example only the response of the output to command signals. This is a common error in the literature.
- The properties of the different transfer functions can be illustrated by their frequency or time responses.

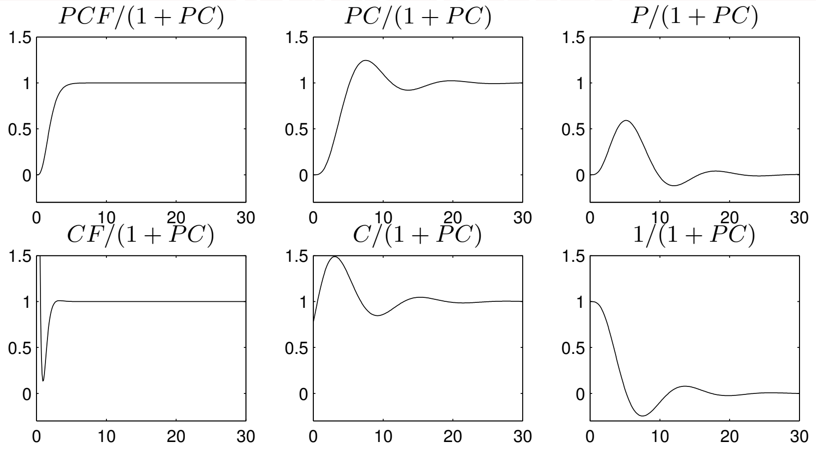
Example: Frequency Responses

PI control ($K = 0.775$, $T_i = 2.05$) of $P(s) = (s+1)^{-4}$ with $G_{yr}(s) = (0.5s+1)^{-4}$. Gain curves:



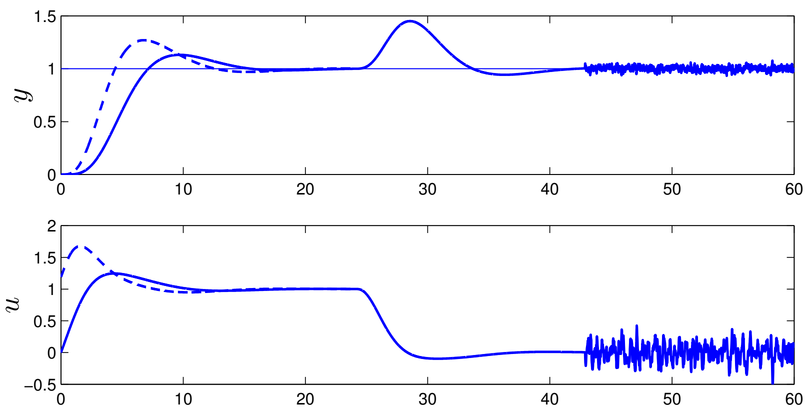
Example: Time Responses

PI control ($K = 0.775$, $T_i = 2.05$) of $P(s) = (s+1)^{-4}$ with $G_{yr}(s) = (0.5s+1)^{-4}$. Step responses:



Time responses – an alternative

Responses to setpoint change, step load disturbance and measurement noise:

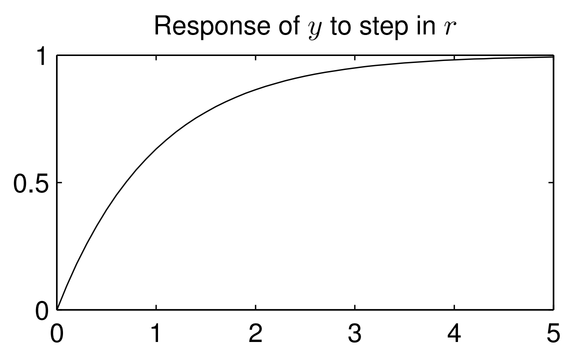


Error feedback (dashed), 2-DOF controller (full)

One plot gives a good overview!

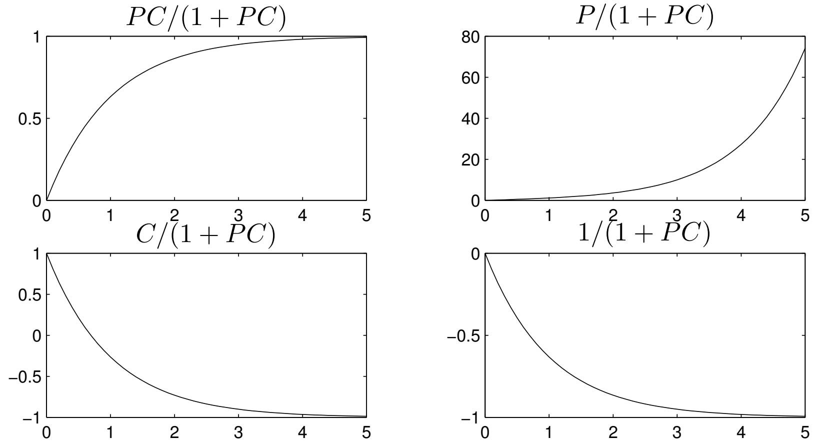
A warning

Remember to always look at **all** responses when you are dealing with control systems. The step response below looks fine, but . . .



Gang of Four

Step responses:



Unstable output response to load disturbance. What is going on?

The system

Process $P(s) = \dfrac{1}{s-1}$

Controller $C(s) = \dfrac{s-1}{s}$

Response of y to setpoint r

$$G_{yr}(s) = \frac{PC}{1+PC} = \frac{1}{s+1}$$

Response of y to step in disturbance d

$$G_{yd}(s) = \frac{P}{1+PC} = \frac{s}{s^2-1} = \frac{s}{(s+1)(s-1)}$$

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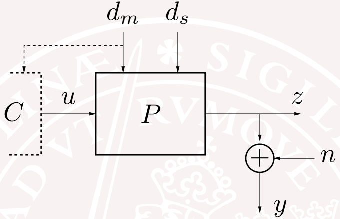
Two kinds of disturbances



Load disturbances d

- Disturbances that affect the process state
  - d_m measurable, can use feedforward
  - d_s non-measurable, must use feedback. Controller should have **high gain** at the dominant frequencies to supress them

Measurement disturbances n

- Disturbances that corrupt the feedback signal(s)
  - Controller should have **low gain** at the dominant frequencies to avoid being “fooled”

Disturbance models

Deterministic disturbance models, e.g., impulse, step, ramp, sinusoidal signals

- Can be modeled by Dirac impulse filtered through linear system

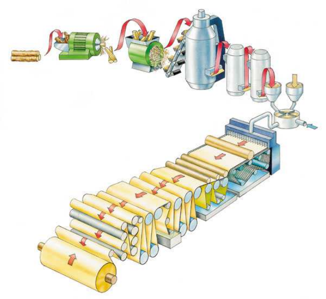
Stochastic disturbance models

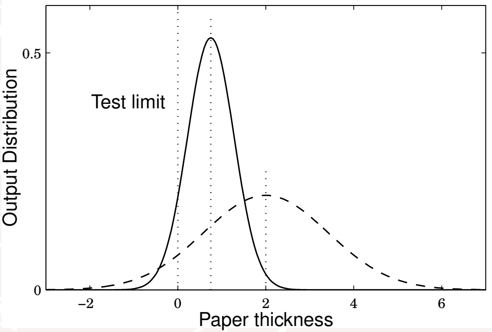
- Common model: Gaussian stochastic process
  - Can be modeled by white noise filtered through linear system
  - Reasonable model for many real-world random fluctuations

Example: control of paper thickness

Control of paper thickness – want to keep down variation in output!

Random process variations modeled as Gaussian load disturbance





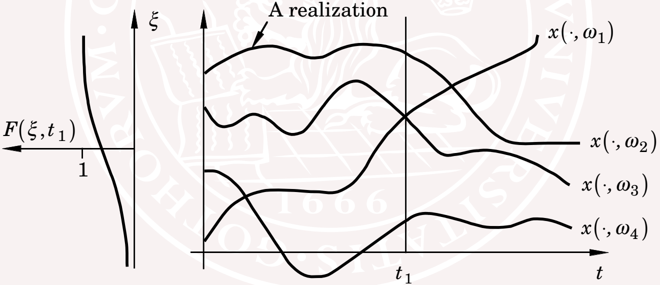
All paper production below the test limit is wasted.
Good control allows for lower setpoint with the same waste. The average thickness is lower, which saves significant costs.

Stochastic process – definition

A **stochastic process** (random process) is a family of stochastic variables $\{x(t), t \in T\}$

Can be viewed as a function of two variables, $x = x(t, \omega)$:

- Fixed $\omega = \omega_0$ gives a time function $x(\cdot, \omega_0)$ (realization)
- Fixed $t = t_1$ gives a random variable $x(t_1, \cdot)$ (distribution)



For a **Gaussian process**, $x(t_1, \cdot)$ has a normal distribution

Gaussian processes

We will work with **zero-mean stationary Gaussian processes**

Mean-value function:

$$m_x = \mathbf{E}x(t) \equiv 0$$

Covariance function:

$$r_x(\tau) = \mathbf{E}x(t+\tau)x(t)^T$$

Cross-covariance function:

$$r_{xy}(\tau) = \mathbf{E}x(t+\tau)y(t)^T$$

A zero-mean stationary Gaussian process x is completely characterized by its covariance function.

Spectral density

The **spectral density** or **spectrum** of a stationary stochastic process is defined as the Fourier transform of the covariance function:

$$\Phi_x(\omega) := \int_{-\infty}^{\infty} r_x(t) e^{-i\omega t} dt$$

Then, by inverse Fourier transform

$$r_x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} \Phi_x(\omega)\, d\omega$$

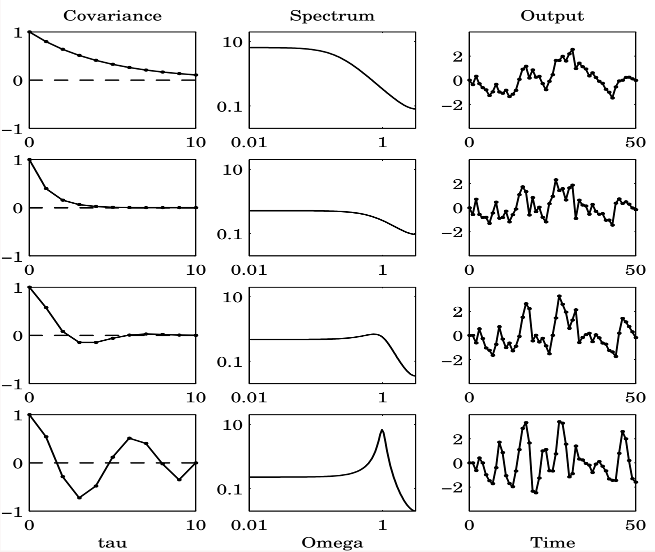
In particular, the **stationary variance** is given by

$$\mathbf{E} x(t) x^T(t) = r_x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega)\, d\omega$$

White noise with **intensity** R means a process v with spectrum

$$\Phi_v(\omega) = R \quad \text{for all } \omega$$

Covariance fcn, spectral density, and sample realization



Error correction: The spectra should be divided by 2π

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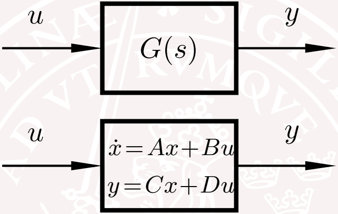
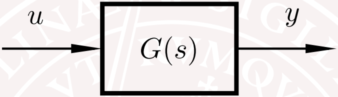
Filtering of white noise



Assume u white noise with intensity R. Two different problems:

1. Given $G(s)$ (or (A, B, C, D)), calculate the spectral density and/or stationary variance of y (or x)
2. Conversely, given the spectral density of y, determine $G(s)$
   - **spectral factorization**

Calculation of spectral density



Assume that u has the spectral density $\Phi_u(\omega)$. Then y gets the spectral density

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G^*(i\omega)$$

Special case: If u is white noise with intensity R, then

$$\Phi_y(\omega) = G(i\omega)RG^*(i\omega)$$

Spectral density for state-space system

Consider the linear system

$$\dot{x} = Ax + Bv, \qquad \Phi_v(\omega) = R$$

The transfer function from v to x is

$$G(s) = (sI - A)^{-1}B$$

and the spectrum for x will be

$$\Phi_x(\omega) = (i\omega I - A)^{-1}BR\,\underbrace{B^*(-i\omega I - A)^{-T}}_{G^*(i\omega)}$$

Stationary covariance of x:

$$\Pi_x = R_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega)d\omega$$

Calculation of stationary state covariance

Theorem [G&L 5.3]

If all eigenvalues of A are strictly in the left half plane then there exists a unique matrix $\Pi_x = \Pi_x^T > 0$ which is the solution to the Lyuapunov equation

$$A\Pi_x + \Pi_x A^T + BRB^T = 0$$

Calculation of state covariance – example

Consider the system

$$\dot{x} = Ax + Bv \quad = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

where v is white noise with intensity 1.

What is the stationary covariance of x?

First check the eigenvalues of A : $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \in LHP$. OK!

Solve the Lyapunov equation $A\Pi_x + \Pi_x A^T + BRB^T = 0_{2,2}$.

Calculation of state covariance – example

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Example cont'd

$$A\Pi_x + \Pi_x A^T + BRB^T = 0_{2\times 2}$$

Find Π_x:

$$\begin{bmatrix}-1 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12} & \Pi_{22}\end{bmatrix} + \begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12} & \Pi_{22}\end{bmatrix}\begin{bmatrix}-1 & -1\\ 2 & 0\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} =$$

$$= \begin{bmatrix}2(-\Pi_{11} + 2\Pi_{12}) + 1 & -\Pi_{12} + 2\Pi_{22} - \Pi_{11}\\ -\Pi_{12} + 2\Pi_{22} - \Pi_{11} & -2\Pi_{12}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Solving for Π_{11}, Π_{12} and Π_{22} gives

$$\Pi_x = \begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12} & \Pi_{22}\end{bmatrix} = \begin{bmatrix}1/2 & 0\\ 0 & 1/4\end{bmatrix} > 0$$

Matlab: `lyap([-1 2; -1 0], [1; 0]*[1 0])`

Example cont'd

$$A\Pi_x + \Pi_x A^T + BRB^T = 0_{2\times 2}$$

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Example cont'd

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Matlab: `lyap([-1 2; -1 0], [1; 0]*[1 0])`

Spectral Factorization

Theorem [G&L 5.1]

Assume that the real valued, scalar function $\Phi_v(\omega) \geq 0$ is a rational function of ω^2, finite for all ω. There is then a rational function $G(s)$, with real coefficients, and with all poles strictly in the left half plane, and all zeros in the left half plane or on the imaginary axis, such that

$$\Phi_v(\omega) = |G(i\omega)|^2 = G(i\omega)G(-i\omega)$$

Spectral Factorization — Example

Find a stable, minimum-phase filter $G(s)$ such that a process y generated by filtering unit intensity white noise through G gives

$$\Phi_y(\omega) = \frac{\omega^2 + 4}{\omega^4 + 10\omega^2 + 9},$$

Solution. We have

$$\Phi_y(\omega) = \frac{\omega^2 + 4}{(\omega^2 + 1)(\omega^2 + 9)} = \left| \frac{i\omega + 2}{(i\omega + 1)(i\omega + 3)} \right|^2$$

implying

$$G(s) = \frac{s + 2}{(s + 1)(s + 3)}$$