FRTN10 Multivariable Control, Lecture 2

Automatic Control LTH, 2016

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

1. Introduction
2. **Stability and robustness**
3. Specifications and disturbance models
4. Control synthesis in frequency domain
5. Case study

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Lecture 2 – Outline

1. Stability
2. Sensitivity and robustness
3. The Small Gain Theorem
4. Singular values

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Stability is crucial

- bicycle
- JAS 39 Gripen
- Mercedes A-class
- ABS brakes

Input–output stability

A system is called **input–output stable** (or "$\mathcal{L}_2$ stable" or just "stable") if its $\mathcal{L}_2$ gain is finite:

$$\|\mathcal{S}\| = \sup_u \frac{\|\mathcal{S}(u)\|_2}{\|u\|_2} < \infty$$

Input–output stability of LTI systems

For an LTI system $\mathcal{S}$ with impulse response $g(t)$ and transfer function $G(s)$, the following stability conditions are equivalent:

- $\|\mathcal{S}\|$ is bounded
- $g(t)$ decays exponentially
- $\int_0^\infty |g(t)|dt$ is bounded
- All poles of $G(s)$ have negative real part

Internal stability

The autonomous LTI system

$$\frac{dx}{dt} = Ax$$

is called **exponentially stable** if the following equivalent conditions hold:

- There exist constants $\alpha, \beta > 0$ such that

$$|x(t)| \leq \alpha e^{-\beta t} |x(0)| \qquad \text{for } t \geq 0$$

- All eigenvalues of A have negative real part

(Exponential stability is a stronger form of asymptotic stability. For LTI systems, they are equivalent.)

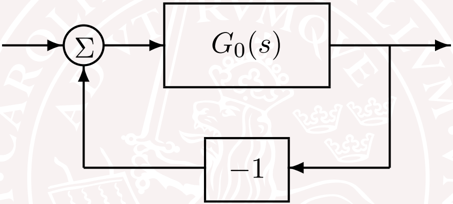
Internal vs input–output stability

If $\dot{x} = Ax$ is exponentially stable **then** $G(s) = C(sI - A)^{-1}B + D$ is input–output stable.

Warning: The opposite is not always true! There may be unstable pole-zero cancellations (that also render the system uncontrollable and/or unobservable), and these may not be seen in the transfer function!

Stability of feedback loops

Assume scalar open-loop system $G_0(s)$



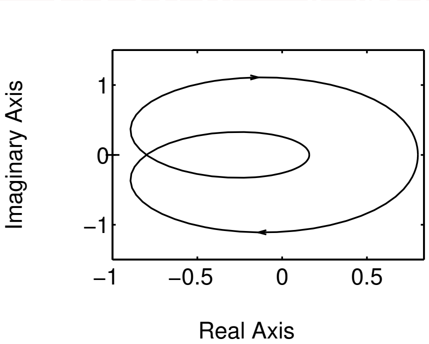
The closed-loop system is input–output stable **if and only if** all solutions to the characteristic equation

$$1 + G_0(s) = 0$$

are in the left half plane (i.e., have negative real part).

The Nyquist criterion

If $G_0(s)$ is stable, then the closed-loop system $[1 + G_0(s)]^{-1}$ is stable **if and only if** the Nyquist curve does not encircle -1.



(Note: Matlab gives a Nyquist plot for both positive and negative frequencies)

The Nyquist criterion (cont'd)

More generally, the difference between the number of unstable poles in $[1 + G_0(s)]^{-1}$ and the number of unstable poles in $G_0(s)$ is equal to the number of times the point -1 is encircled by the Nyquist plot in the clockwise direction.

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Sensitivity and robustness

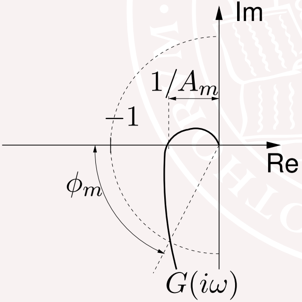
- How sensitive is the closed-loop system to model errors?
- How do we measure the “distance to instability”?
- Is it possible to guarantee stability for all systems within some distance from the ideal model?

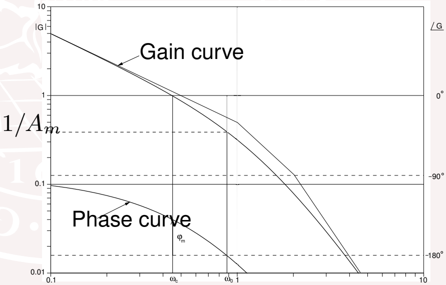
Amplitude margin A_m:

$$\arg G(i\omega_0) = -180^\circ, \quad |G(i\omega_0)| = \frac{1}{A_m}$$

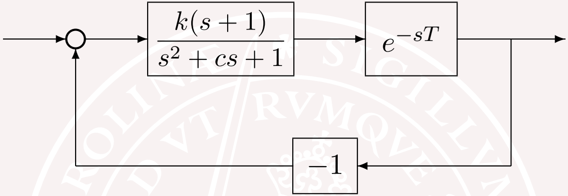
Phase margin ϕ_m:

$$|G(i\omega_c)| = 1, \quad \arg G(i\omega_c) = \phi_m - 180^\circ$$

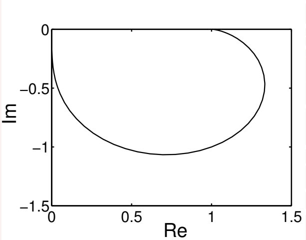


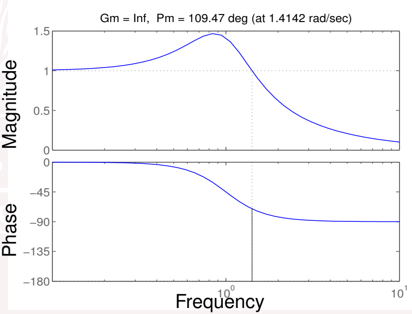


Mini-problem

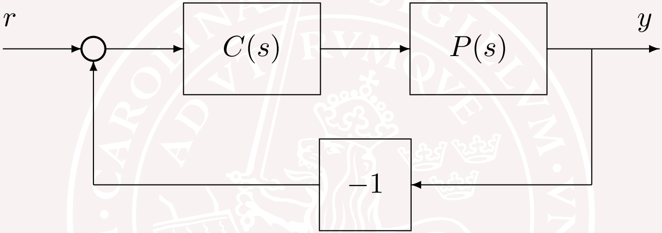


Nominally $k = 1$, $c = 1$ and $T = 0$. How much margin is there in each of the parameters before the system becomes unstable?





How sensitive is the closed loop to changes in the plant?



$$Y(s) = \underbrace{\frac{P(s)C(s)}{1 + P(s)C(s)}}_{T(s)} R(s)$$

$$\frac{dT}{dP} = \frac{d}{dP}\left(1 - \frac{1}{1+PC}\right) = \frac{C}{(1+PC)^2} = \frac{T}{P(1+PC)}$$

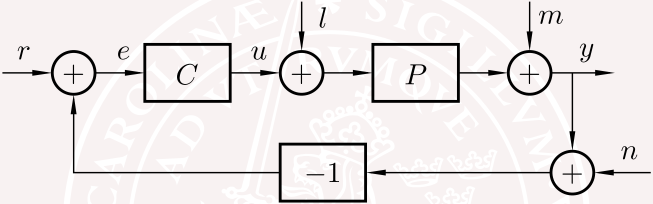
Define the **sensitivity function**, S,

$$S := \frac{d(\log T)}{d(\log P)} = \frac{dT/T}{dP/P} = \frac{1}{1+PC}$$

and the **complementary sensitivity function** T,

$$T := 1 - S = \frac{PC}{1+PC}$$

Interpretation as disturbance sensitivities



Note that

- $T = -G_{yn}$ (sensitivity towards measurement noise)
- $S = G_{ym}$ (sensitivity towards output load disturbance)

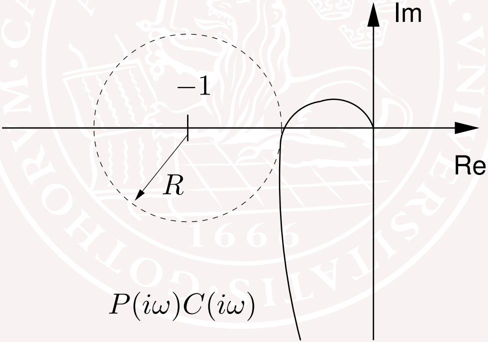
Fundamental limitation:

$$S + T = 1$$

Interpretation as stability margin

The sensitivity function measures the distance between the Nyquist plot and the point -1:

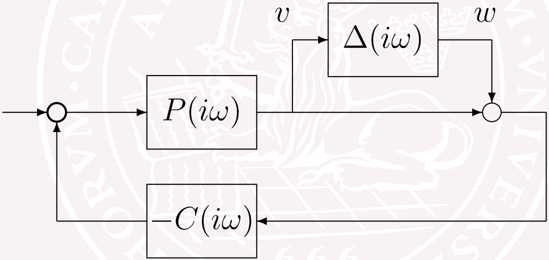
$$R^{-1} = \sup_{\omega} \left| \frac{1}{1 + P(i\omega)C(i\omega)} \right| = M_s$$



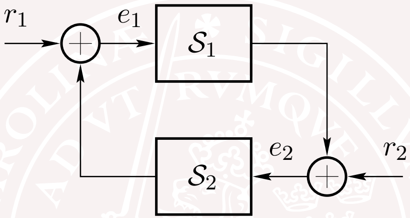
Lecture 2 – Outline

Robustness analysis

How large plant uncertainty $\Delta(i\omega)$ can be tolerated without risking instability?



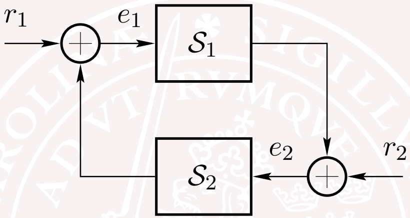
The Small Gain Theorem



Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are input-output stable. **If** $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$, **then** the gain from (r_1, r_2) to (e_1, e_2) in the closed-loop system is finite.

- Note 1: The theorem applies also to nonlinear, time-varying, and multivariable systems
- Note 2: The stability condition is sufficient but not necessary, so the results may be conservative

The Small Gain Theorem



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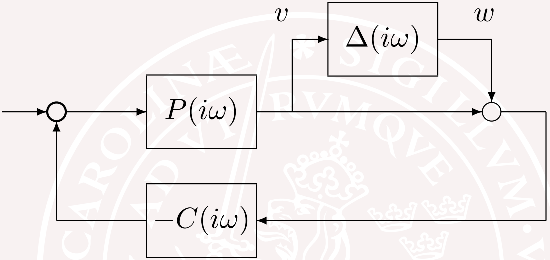
Proof sketch

$$
\begin{aligned}
e_1 &= r_1 + \mathcal{S}_2(r_2 + \mathcal{S}_1(e_1)) \\
\|e_1\| &\le \|r_1\| + \|\mathcal{S}_2\|\Big(\|r_2\| + \|\mathcal{S}_1\| \cdot \|e_1\|\Big) \\
\|e_1\| &\le \frac{\|r_1\| + \|\mathcal{S}_2\| \cdot \|r_2\|}{1 - \|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\|}
\end{aligned}
$$

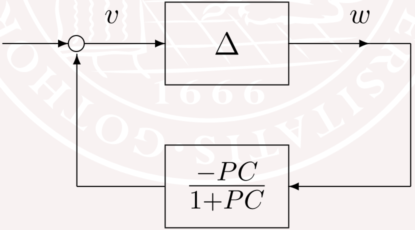
This shows bounded gain from (r_1, r_2) to e_1.

The gain to e_2 is bounded in the same way.

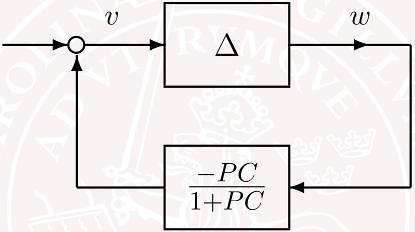
Application to robustness analysis



The diagram can be redrawn as



Application to robustness analysis



The Small Gain Theorem guarantees stability if

$$\|\Delta(i\omega)\|_\infty \cdot \left\| \frac{P(i\omega)C(i\omega)}{1+P(i\omega)C(i\omega)} \right\|_\infty < 1$$

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Gain of multivariable systems

Recall from Lecture 1 that

$$||\mathcal{S}|| = \sup_{\omega} |G(i\omega)| = ||G||_{\infty}$$

for a stable LTI system $\mathcal{S}$.

How to calculate $|G(i\omega)|$ for a multivariable system?

Vector norm and matrix gain

For a vector $x \in \mathbf{C}^n$, we use the 2-norm

$$|x| = \sqrt{x^* x} = \sqrt{|x_1|^2 + \cdots + |x_n|^2}$$

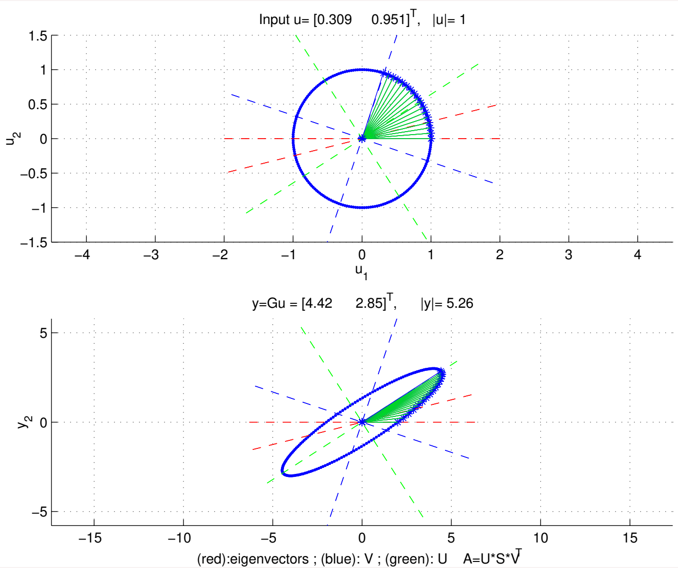
For a matrix $A \in \mathbf{C}^{n\times m}$, we use the L_2-induced norm

$$\|A\| := \sup_x \frac{|Ax|}{|x|} = \sup_x \sqrt{\frac{x^* A^* A x}{x^* x}} = \sqrt{\bar{\lambda}(A^* A)}$$

$\bar{\lambda}(A^* A)$ denotes the largest eigenvalue of $A^* A$. The ratio $|Ax|/|x|$ is maximized when x is a corresponding eigenvector.

(A^* denotes the **conjugate transpose** of A)

Example: Different gains in different directions: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



Singular Values

For a matrix A, its singular values σ_i are defined as

$$\sigma_i = \sqrt{\lambda_i}$$

where λ_i are the eigenvalues of A^*A.

Let $\bar{\sigma}(A)$ denote the largest singular value and $\underline{\sigma}(A)$ the smallest singular value.

For a linear map $y = Au$, it holds that

$$\underline{\sigma}(A) \leq \frac{|y|}{|u|} \leq \bar{\sigma}(A)$$

The singular values are typically computed using singular value decomposition (SVD):

$$A = U\Sigma V^*$$

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SVD example

Matlab code for singular value decomposition of the matrix

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix}$$

SVD:

$$A = U \cdot S \cdot V^*$$

where both the matrices U and V are unitary (i.e. have orthonormal columns s.t. $V^* \cdot V = I$) and S is the diagonal matrix with (sorted decreasing) singular values σ_i.
Multiplying A with a input vector along the first column in V gives

$$A \cdot V_{(:,1)} = USV^* \cdot V_{(:,1)} =$$

$$= US \begin{bmatrix} 1 \\ 0 \end{bmatrix} = U_{(:,1)} \cdot \sigma_1$$

That is, we get maximal gain σ_1 in the output direction $U_{(:,1)}$ if we use an input in direction $V_{(:,1)}$ (and minimal gain $\sigma_n = \sigma_2$ if we use the last column $V_{(:,n)} = V_{(:,2)}$).

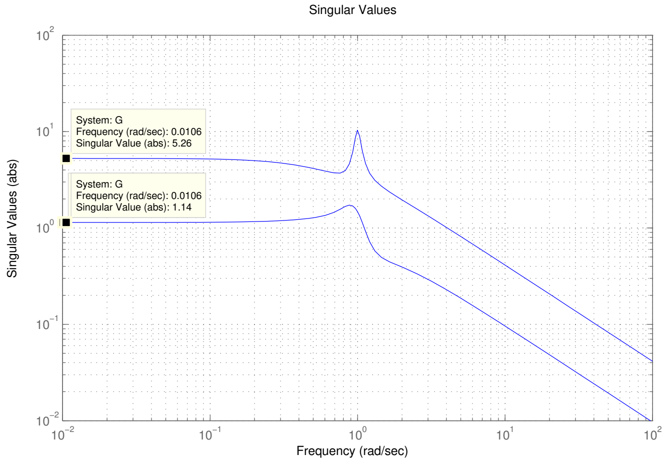
```
>> A=[2 4 ; 0 3]
A =
     2     4
     0     3
>> [U,S,V]=svd(A)
U =
    0.8416   -0.5401
    0.5401    0.8416
S =
    5.2631         0
         0    1.1400
V =
    0.3198   -0.9475
    0.9475    0.3198

>> A*V(:,1)
ans =
    4.4296
    2.8424

>> U(:,1)*S(1,1)
ans =
    4.4296
    2.8424
```

Example: Gain of multivariable system

Consider the transfer function matrix

$$G(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{4}{2s+1} \\ \dfrac{s}{s^2+0.1s+1} & \dfrac{3}{s+1} \end{bmatrix}$$

```
>> s=tf('s')
>> G=[ 2/(s+1) 4/(2*s+1); s/(s^2+0.1*s+1) 3/(s+1)];
>> sigma(G) % plot sigma values of G  wrt fq
>> grid on
>> norm(G,inf)  % infinity norm = system gain
   ans =
      10.3577
```



The singular values of the tranfer function matrix (prev slide). Note that $G(0)$= [2 4 ; 0 3] which corresponds to A in the SVD-example above.
$\|G\|_\infty = 10.3577$.