

FRTN10 Exercise 13. Controller Simplification

13.1 Consider a SISO system for which the pole-zero map is given in Figure 13.1.

- Determine the transfer function of the system. You can assume that the static gain is $G(0) = 1$.
- By studying the pole-zero map, it is possible to get a hint that the system is a candidate for model order reduction. How?
- Use a computer to calculate a balanced realization and the Hankel singular values of the system. Perform a model reduction by eliminating the state corresponding to the smallest singular value.

Useful commands: balreal, modred.

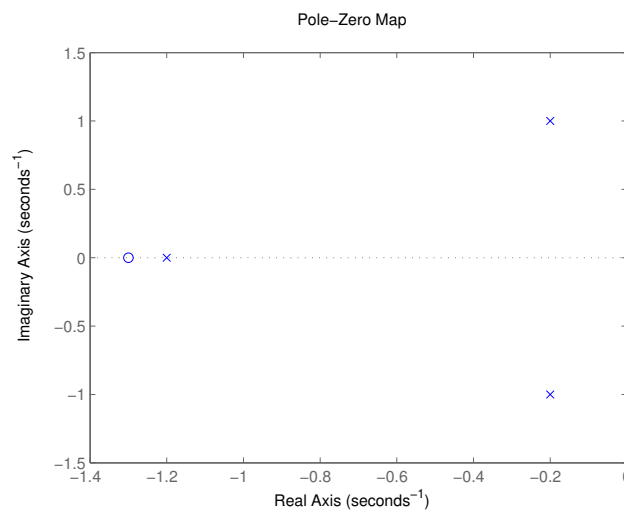


Figure 13.1 Pole-zero map of the system in Problem 13.1

13.2 For the system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ -1 & -0.5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 10u$$

solve the following problems by hand:

- Verify that the controllability gramian is $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ while $\begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$ is the observability gramian.
- Determine the Hankel singular values.
- Find a coordinate change that gives a balanced realization.
- Find a reduced system $G_1(s)$ by truncating the state corresponding to the smallest Hankel singular value.

Exercise 13. Controller Simplification

13.3 For the same system and notation as in the previous problem, use a computer for the following:

- a. Find the transfer function $G(s)$ from u to y .
- b. Compare the error $\max_{\omega} |G(i\omega) - G_1(i\omega)|$ with the error bound for balanced truncation.
- c. Find a reduced system G_2 by truncating both states and keeping just a constant gain.
- d. Compare the error $\max_{\omega} |G(i\omega) - G_2(i\omega)|$ with the error bound for balanced truncation.

13.4 Find a reduced order approximation of

$$\frac{2s^2 + 2.99s + 1}{s(s + 1)^2}$$

by writing the transfer function as the sum of an integrator and a stable transfer function, then applying balanced truncation to the stable part. You may use a computer.

Solutions to Exercise 13. Controller Simplification

13.1 a. Inspection of the locations of the poles and zeros gives us the transfer function

$$G(s) = 1.04 \frac{s/1.3 + 1}{(s/1.2 + 1)(s^2 + 0.4s + 1.04)}$$

- b.** The closeness of the pole-zero pair on the real axis suggests that a model reduction might be possible.
- c.** A balanced realization and the Hankel singular values for the system can be calculated using the Matlab command

```
>>> s = tf('s');
>>> G = 1.04*(s/1.3+1)/((s/1.2+1)*(s^2+0.4*s+1.04));
>>> [balr,g] = balreal(G);
```

which gives the following Hankel singular values:

$$g = \begin{pmatrix} 1.5105 \\ 1.0196 \\ 0.0091 \end{pmatrix}$$

Elimination of the state in the balanced realization corresponding to the smallest Hankel singular value is done in Matlab by

```
>>> modsys = modred(balr,g<0.01)
>>> modsysG = tf(modsys)
```

This gives the following transfer function for the reduced order system:

$$G_{red}(s) = 0.0181 \frac{s^2 - 2.412s + 57.49}{s^2 + 0.4086s + 1.043}$$

A Bode magnitude plot of the original system and the reduced system is shown in figure 13.1.

13.2 a. With

$$S = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 0 \\ -1 & -0.5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

we have

$$AS + SA^T + BB^T = \begin{pmatrix} -2 & 0 \\ -2 & -0.5 \end{pmatrix} + \begin{pmatrix} -2 & -2 \\ 0 & -0.5 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so S is the controllability gramian. Similarly, with

$$O = \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix}$$

$$OA + A^T O + C^T C = \begin{pmatrix} -0.5 & 0 \\ -1 & -0.5 \end{pmatrix} + \begin{pmatrix} -0.5 & -1 \\ 0 & -0.5 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

so O is the observability gramian.

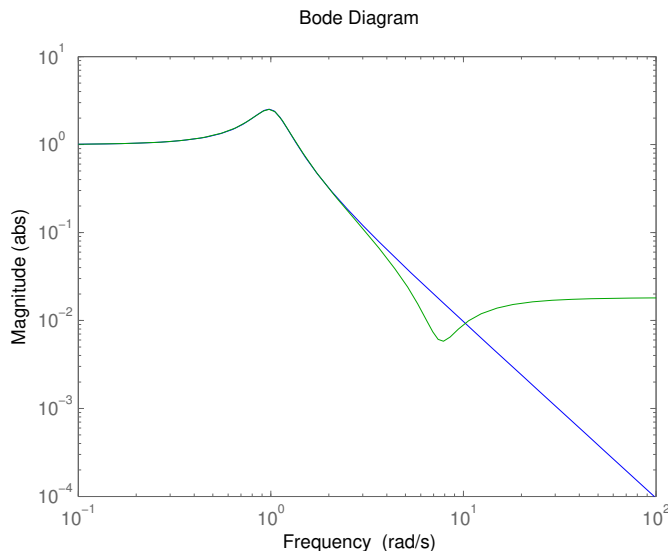


Figure 13.1 Bode magnitude plot of the original and reduced system in Problem 13.1

- b.** The Hankel singular values are the eigenvalues of

$$SO = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so they are both 1.

- c.** The coordinate change $\xi = Tx$ yields the new gramians $S_\xi = TST^T$ and $O_\xi = T^{-T}OT^{-1}$. To find T we solve the equation $S_\xi = O_\xi$. Since both S and O are diagonal it seems reasonable that a diagonal T will work. With $T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}$ we get the equations

$$TST^T = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} = \begin{pmatrix} 2t_1^2 & 0 \\ 0 & t_2^2 \end{pmatrix}$$

and

$$T^{-T}OT^{-1} = \begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix} \begin{pmatrix} 0.5 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/t_1 & 0 \\ 0 & 1/t_2 \end{pmatrix} = \begin{pmatrix} 0.5/t_1^2 & 0 \\ 0 & 1/t_2^2 \end{pmatrix}$$

which gives

$$\begin{aligned} 2t_1^2 &= 0.5/t_1^2 \Rightarrow t_1^4 = 1/4 \Rightarrow t_1 = 1/\sqrt{2} \\ t_2^2 &= 1/t_2^2 \Rightarrow t_2^4 = 1 \Rightarrow t_2 = 1 \end{aligned}$$

(You could also use the direct formula for T in the proof on page 81 in [Glad & Ljung])

With this T

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{pmatrix}$$

the gramians become

$$S_\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad O_\xi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, a balanced realization is

$$\begin{aligned}\dot{\xi} &= \hat{A}\xi + \hat{B}u \\ y &= \hat{C}\xi + \hat{D}u\end{aligned}$$

where

$$\begin{aligned}\hat{A} &= TAT^{-1} = \begin{pmatrix} -1 & 0 \\ -\sqrt{2} & -0.5 \end{pmatrix} & \hat{B} &= TB = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} \\ \hat{C} &= CT^{-1} = (\sqrt{2} \quad 1) & \hat{D} &= D\end{aligned}$$

- d.** In this case, the Hankel singular values have the same size, therefore either could be removed. (However, this means that it is probably not a good idea to do any truncation at all!) If the second state is removed by letting $\xi_2 = 0$, ξ_2 can be expressed in terms of ξ_1 through $0 = \hat{A}_{21}\xi_1 + \hat{A}_{22}\xi_2 + \hat{B}_2u$. The reduced realization then becomes

$$\begin{aligned}\dot{\xi}_1 &= (\hat{A}_{11} - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{A}_{21})\xi_1 + (\hat{B}_1 - \hat{A}_{12}\hat{A}_{22}^{-1}\hat{B}_2)u \\ y_r &= (\hat{C}_1 - \hat{C}_2\hat{A}_{22}^{-1}\hat{A}_{21})\xi_1 + (\hat{D} - \hat{C}_2\hat{A}_{22}^{-1}\hat{B}_2)u\end{aligned}$$

where for example $\hat{A}_{21}$ is the element in the second row and first column in $\hat{A}$.

$$\begin{aligned}\dot{\xi}_1 &= -\xi_1 + \sqrt{2}u \\ y_r &= -\sqrt{2}\xi_1 + 12u\end{aligned}$$

The transfer function is obtained through the Laplace transform

$$G_1(s) = 12 - \frac{2}{s+1}$$

13.3 a. The Matlab command `tf(ss(A,B,C,D))` gives

$$G(s) = \frac{10s^2 + 18s + 5}{s^2 + 1.5s + 0.5}$$

- b.** Plotting the Bode diagram for $G(s) - G_1(s)$ through the command `bodemag(G-G1)` gives 2 as the maximal error, obtained at large frequencies. The error bound, twice the sum of the truncated singular values, also gives 2. In this case the error bound is tight.
- c.** Truncating both states gives

$$G_2 = \hat{D} - \hat{C}\hat{A}^{-1}\hat{B} = 10$$

- d.** Plotting `bodemag(G-Gr)` gives 2 as the maximal error, near $\omega = 1$. The error bound $2(1+1) = 4$ is conservative.

13.4 Through partial fractions one can write

$$\frac{2s^2 + 2.99s + 1}{s(s+1)^2} = \frac{1}{s} + \frac{s+0.99}{(s+1)^2}$$

The Matlab command

`[G3bal,sig]=balreal(tf([1 .99],[1 2 1]))` gives

$$sig = \begin{pmatrix} 0.4950 \\ 0.00001 \end{pmatrix}$$

so one state can be removed right away.

`G3red=modred(G3bal,(sig<0.1))` yields

$$\frac{-2.525 \cdot 10^{-5}s + 1}{s + 1.01} \approx \frac{1}{s + 1.01}$$

With the integrator we get the reduced system

$$\frac{1}{s} + \frac{1}{s + 1.01} = \frac{2s + 1.01}{s(s + 1.01)}$$

The commands `balreal` and `modred` can actually be used directly on systems with an integrator since they do the separation automatically.