FRTN10 Exercise 10. LQG, Preparations for Lab 3

10.1 Consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 & 6 \\ 0 & 4 \end{pmatrix} u + v_1 \\ y &= \begin{pmatrix} 1 & 1 \end{pmatrix} x + v_2 \\ z &= \begin{pmatrix} 1 & 1 \end{pmatrix} x\end{aligned}$$

a. Design an LQG controller for the system, assume initially that process and measurement noise are independent and have intensity 1, and that we should weight the control signals u and output z exactly the same.

Useful commands: `lqry, kalman, lqgreg`

b. Using the states x and $\hat{x}$, write the closed-loop system in state-space form using symbols. Use L for state-feedback gain and K for Kalman filter gain.

c. Simulate the system without noise with the initial state $x = \begin{pmatrix} 1 & -1 \end{pmatrix}^T$. Plot both process states and estimated states. The Kalman filter begins with its estimates in 0. Try different noise intensities, any conclusions?

Useful commands: `lqgreg, feedback, initial`

10.2 Consider the problem of controlling a double integrator

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + v_1$$

where the white noise v_1 has intensity I. We can only measure x_1, unfortunately with added white noise also of intensity 1. We want to minimize the cost

$$J = \int_0^\infty \left(x_1^2 + x_2^2 + u^2\right) dt.$$

Solve the control problem by hand (not using Matlab) and give the controller in state-space form.

10.3 Do the preparatory exercises for Laboratory Session 3. The lab manual is found on the course homepage.

Solutions to Exercise 10. LQG, Preparations for Lab 3

10.1 a. See **c**.

b. Using the state vector $x_e = \begin{pmatrix} x^T & \hat{x}^T \end{pmatrix}^T$ and the obvious notation A, B, C, we get the system

$$\dot{x}_e = \begin{pmatrix} A & -BL \\ KC & A - BL - KC \end{pmatrix} x_e + \begin{pmatrix} I \\ 0 \end{pmatrix} v_1 + \begin{pmatrix} 0 \\ K \end{pmatrix} v_2$$
$$z = \begin{pmatrix} C & 0 \end{pmatrix} x_e$$

c. With less measurement noise the estimated states converge faster to the actual states, and the output z converge faster to zero. See Figures 10.1-10.2 and Matlab code below.

As shown in exercise 9.1, only the relation between process noise and measurement noise matters. More process noise will therefore have the same effect as less measurement noise.

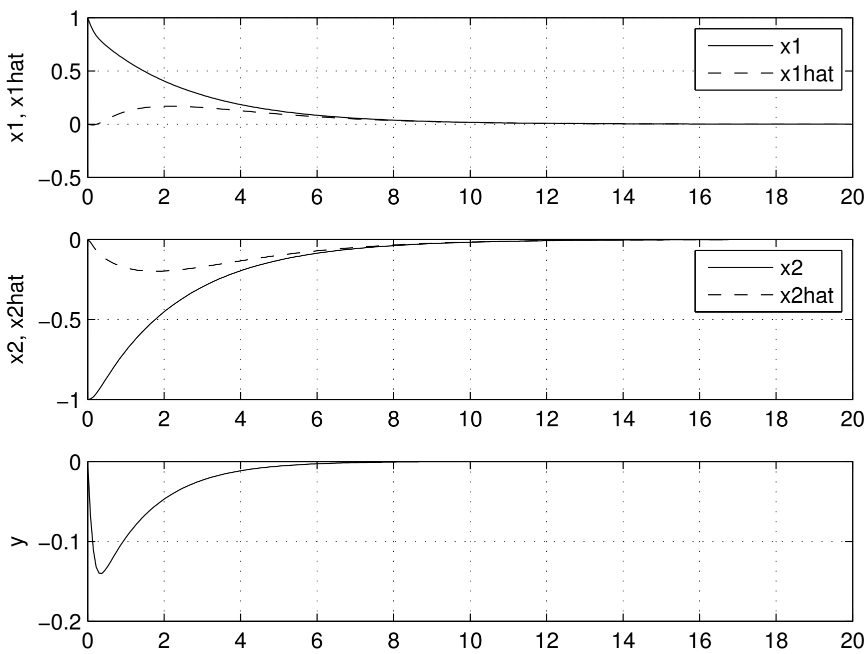

Figure 10.1 Initial response if little measurement noise ($\frac{R2}{R1} = 0.1$).

```
A = [0 1; 0 0];
B = [1 6; 0 4];
C = [1 1];
D = zeros(1,2);

% State feed-back design
```

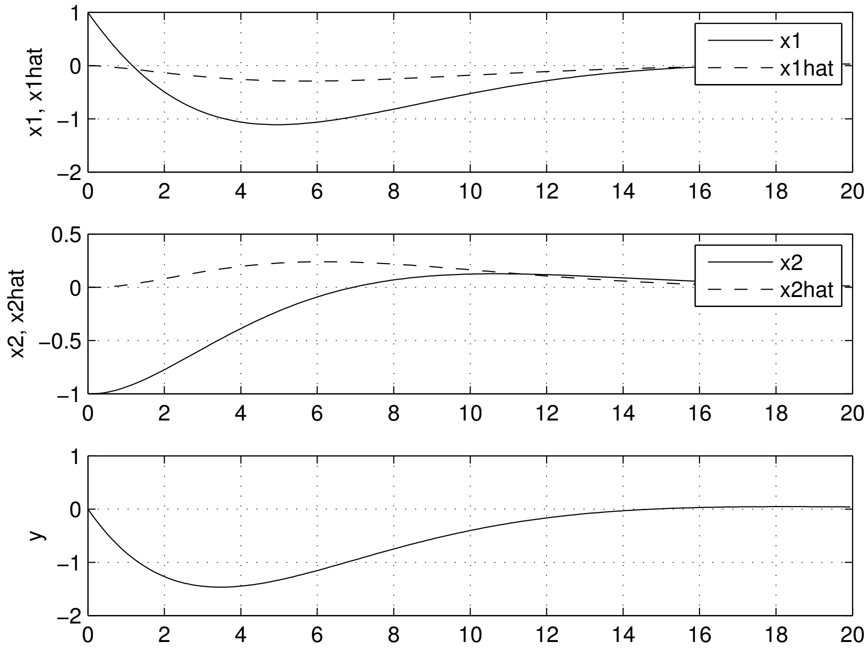

Figure 10.2 Initial response if much measurement noise ($\frac{R2}{R1} = 100$).

```
process = ss(A,B,C,D);
Q1 = 1;
Q2 = eye(2);
[L,S,E] = lqry(process,Q1,Q2);

% Kalman filter design
G = eye(2);
H =zeros(1,2);
syskalman = ss(A,[B G],C,[D H]);
R1 = eye(2);
R2 = 1;
[Kest,K,E] = kalman(syskalman,R1,R2);

% Construct closed loop
reg = lqgreg(Kest,L);
closed_loop = feedback(process,-reg);

% Plot response
[Y,T,X] = initial(closed_loop,[1 -1 0 0],0:0.01:20);
subplot(311)
plot(T, X(:,1)); hold on; plot(T, X(:,3),'--'); grid
legend('x1','x1hat'); ylabel('x1, x1hat')
subplot(312)
plot(T, X(:,2)); hold on; plot(T, X(:,4),'--'); grid
legend('x2','x2hat'); ylabel('x2, x2hat')
subplot(313)
plot(T,Y); grid; ylabel('y');
```

10.2 First of all, we see that we can not measure the states we want to control, so we need a Kalman filter. We start by setting up the problem in the standard form

$$\begin{aligned}\dot{x} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + v_1 \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + v_2 \\ z &= x\end{aligned}$$

where v_2 is white noise with intensity 1. The cost function is

$$J = \int_0^\infty \left(z^T Q_1 z + u Q_2 u \right) dt$$

with $Q_1 = I_2$ and $Q_2 = 1$.
For the state feed-back gain, we have to solve the Riccati equation

$$A^T S + SA + Q_1 - SBQ_2^{-1} B^T S = 0$$

This gives the following equations,

$$\begin{aligned} 1 - s_2^2 &= 0 \\ s_1 - s_2 s_3 &= 0 \\ 2s_2 + 1 - s_3^2 &= 0 \end{aligned}$$

with the solution $s_1 = s_3 = \sqrt{3}$, $s_2 = 1$. This gives the state feed-back vector $L = B^T S = \begin{pmatrix} 1 & \sqrt{3} \end{pmatrix}$.
For the Kalman filter we must solve the Riccati equation

$$AP + PA^T + R_1 - PC^T CP^T = 0$$

with $R_1 = I_2$, which gives

$$\begin{aligned} 2p_2 + 1 - p_1^2 &= 0 \\ p_3 - p_1 p_2 &= 0 \\ 1 - p_2^2 &= 0 \end{aligned}$$

Using the solution for S we have that $p_1 = p_3 = \sqrt{3}$ and $p_2 = 1$ and $K = PC^T = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$
The controller is given by

$$\begin{aligned} \dot{\hat{x}} &= (A - BL - KC)\hat{x} + Ky \\ u &= -L\hat{x} \end{aligned}$$

and we have that

$$A - BL - KC = \begin{pmatrix} -\sqrt{3} & 1 \\ -2 & -\sqrt{3} \end{pmatrix}$$

10.3 No solution provided.