FRTN10 Exercise 9. Kalman Filtering, LQG

9.1 Consider the first order unstable system with the dynamics

$$G(s) = \frac{1}{s-1}$$

and with a state-space representation with additive noise

$$\begin{aligned}
\dot{x}(t) &= x(t) + u(t) + v_1(t) \\
z(t) &= x(t) \\
y(t) &= x(t) + v_2(t) \\
\phi_{v_i} &= R_i
\end{aligned}$$

The uncorrelated noise signals $v_i(t)$ are white with intensities R_i. We are about to investigate how the optimal Kalman filter depends on the R_i's.

a. Show that the optimal Kalman filter only depends on the ratio $\beta = R_1/R_2$.

b. Find the error dynamics, i.e., the dynamics of the estimation error $e(t) = x(t) - \hat{x}(t)$.

c. How does the error dynamics depend on the ratio $\beta = R_1/R_2$? Interpret the result for large β (process noise much larger than measurement noise), and for small β (measurement noise much larger than process noise).

9.2 A Kalman filter should be designed for the second order system

$$\begin{aligned}
\dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_1(t) \\
y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v_2(t) \\
\phi_{v_i} &= 1
\end{aligned}$$

where v_i are uncorrelated white noise with intensity 1.

Design the Kalman filter by

a. solving the algebraic Riccati equation by using `care` in Matlab.

b. using `lqe` in Matlab.

9.3 Consider the first-order stable system with dynamics

$$G(s) = \frac{1}{s+1}$$

and a state-space representation with additive noise

$$\begin{aligned}
\dot{x}(t) &= -x(t) + u(t) + v_1(t) \\
z(t) &= x(t) \\
y(t) &= x(t) + v_2(t) \\
\phi_{v_i} &= 1
\end{aligned}$$

The noise signals $v_i(t)$ are uncorrelated. Often, we have load disturbances acting on the system, hence there is a need for integral action for acceptable control. Using LQ-techniques in designing a state-feedback controller does not automatically give integral action. One way to introduce integral action is to model the disturbance as filtered white noise and use a Kalman filter to estimate the disturbance.

The load disturbance is then modelled as a signal w that influences y and z

$$\begin{aligned} z(t) &= x(t) + w(t) \\ y(t) &= x(t) + w(t) + v_2(t) \end{aligned}$$

In order to model the static error in z and y, w should have large low-frequency content. To use a Kalman filter to estimate the error, we need to find a filter $H(s)$ that generates the signal w from a white noise process n

$$w = Hn.$$

For true integral action we want $H(s) = 1/s$, but with this model the noise state will be neither controllable nor stable, and we will not be able to design an LQG controller for the extended system. To get around this problem, we replace the pure integrator by a first order system

$$H(s) = \frac{1}{s+\delta}$$

for some small δ.

a. Find a state-space realization of the extended system, including the noise model

$$\begin{aligned} \dot{x}_e &= A_e x_e + B_e u_e + N_e v_{1e} \\ y &= C_e x_e + v_2 \\ z &= M_e x_e \end{aligned}$$

where $v_{1e} = \begin{pmatrix} v_1 \\ n \end{pmatrix}$

b. Design the full LQG controller in Matlab using the extended model. Note, for large penalties on u the extended noise model doesn't appear to have any effect. Why?

c. Examine the Bode plot of the controller. How does the (almost) integral action in the controller change when changing δ and the noise intensity corresponding to the added state?

d. Will the response to constant load disturbances have a static error?

9.4 Consider control of a DC-motor,

$$G(s) = \frac{1}{s(s+1)}$$

White process noise is active on both states with intensity 1 and with input vector $\begin{pmatrix} 0.1 & 0.1 \end{pmatrix}^T$. There is also noise on the output with intensity 0.1. Let the states be $x_1 = y$, $x_2 = \dot{y}$. This gives the following state-space model

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} v_1(t) \\ y(t) &= \begin{pmatrix} 1 & 0 \end{pmatrix} x(t) + v_2(t) \end{aligned}$$

with $R_1 = 1$, $R_2 = 0.1$ and $R_{12} = 0$

One wishes to use the motor to drive a system that might be oscillatory at the frequency 0.5 rad/s, but there is not much knowledge about its properties.

a. How can you change the model such that the LQG controller will have good robustness at this frequency (a small complementary sensitivity function)? Derive this extended model and determine the intensity matrices needed to solve for the Kalman filter gain.

b. Compute the Kalman filter using `lqe` in Matlab. Plot the transfer function from $y(t)$ to $\hat{y}(t) = C\hat{x}(t)$. Can you see the implication of the noise modelling?

9.5 Consider the task of estimating the states of a double integrator where noise with intensity 1 affects the input only and we have measurement noise of intensity 1.

a. Determine the optimal Kalman filter (by hand calculations).

b. What are the Kalman filter poles?

Solutions to Exercise 9. Kalman Filtering, LQG

9.1 **a.** We have that $A = B = C = N = M = 1$. The Riccati equation thus reduces to

$$2P + R_1 - \frac{P^2}{R_2} = 0,$$

which has the positive semi-definite solution $P = R_2 + R_2\sqrt{1 + \frac{R1}{R2}}$. Thus, the Kalman filter gain is

$$K = \frac{1}{R_2}P = 1 + \sqrt{1 + \frac{R_1}{R_2}} = 1 + \sqrt{1+\beta}.$$

b. The Kalman filter dynamics are given by

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(y(t) - C\hat{x}(t))$$

where $y(t) = Cx(t) + v_2(t)$. Using the values $A = B = C = N = M = 1$ we have the error dynamics

$$\dot{e}(t) = (A-KC)e(t) - Kv_2(t) + v_1(t) = -\sqrt{1+\beta}e(t) - (1+\sqrt{1+\beta})v_2(t) + v_1(t)$$

c. The position of the Kalman filter pole is $-\sqrt{1+\beta}$. We can see that if $\beta \to \infty$, the pole of the Kalmanfilter $\to -\infty$. Hence, the estimation error dynamics are fast, we believe very much in our measurements. On the other hand, if $\beta \to 0$, the Kalman filter pole tends to -1, that is, as fast as the process pole. Now, we trust the model more than the measurements.

9.2 See Matlab code below.

```
>> A = [0 1;1 0];
>> C = [1 0];
>> N = [1 1]';
```

a.

```
>> % Using care
>> Q = N*N';
>> R = 1;
>> S = zeros(2,1);
>> E = eye(2);

>> [X,K,G] = care(A', C', Q, R, S, E);

>> K1 = X*C'

K1 =

    2.4142
    2.4142
```

```
>> eig(A-K1*C)

ans =

   -1.4142
   -1.0000
```

b.

```
>> % Using lqe
>> [K2,P,E] = lqe(A,N,C,1,1,0)

K2 =

    2.4142
    2.4142


P =

    2.4142    2.4142
    2.4142    2.4142


E =

   -1.4142
   -1.0000

>> eig(A-K2*C)

ans =

   -1.4142
   -1.0000
```

9.3 **a.** The noise model has the following state-space realization,

$$\begin{aligned}\dot{x}_w(t) =& -\delta x_w(t) + n(t)\\ w(t) =& x_w(t)\end{aligned}$$

Extending the state-space model of the process with the noise model gives,

$$\begin{aligned}\dot{x}_e(t) &= \begin{pmatrix} -1 & 0 \\ 0 & -\delta \end{pmatrix} x_e(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v_{1e}(t)\\ y(t) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x_e(t) + v_2(t)\\ z(t) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x_e(t)\end{aligned}$$

Note here that $z(t)$ contains the noise state $x_w(t)$, so that, if we design an LQG controller we will try to minimize the disturbance state effect also.

b.

$$
\begin{aligned}
\dot{\hat{x}}(t) &= A_e\hat{x}(t) + B_e u(t) + K(t)(y(t) - C\hat{x}(t)) \\
K(t) &= P(t)C_e^T R_2^{-1} \\
\dot{P}(t) &= A_e P(t) + PA_e^T - K(t)R_2K^T(t) + NR_1N \\
\phi_{v_i} &= 1 \\
u(t) &= -L\hat{x}(t) \\
L &= Q_2^{-1}B_e^T S \\
0 &= A_e^T S + SA + M^T Q_1 M - SB_e Q_2^{-1} B^T S
\end{aligned}
$$

We are looking for the stationary Kalman filter and therefore solve for $\dot{P}(t) = 0$ as before. R_i are noise intensities and Q_i are the weighting matrices for the LQ-problem.

See Matlab code in (**d**). Why do we need a small weight on $u(t)$? Since integral action requires the control signal magnitude to be large at low frequencies we have to let the control signal be large, otherwise the low frequency gain will be limited independent of noise model.

c. If we change the cut-off frequency of the noise filter, we change the cut-off frequency of the low frequency gain of the controller, this is shown in Figure 9.1.

If we on the other hand change the noise intensity, we indirectly change the gain of the noise filter. Hence, we will increase the controller gain for all frequencies, see Figure 9.2.

d. Response to constant load disturbances will *always* have a static error since we do not have infinite gain at low frequencies. That is, we do not have pure integral action, only approximative.

See below for Matlab code,

```
B = [1; 0];
C = [1 1];
D = 0;
N = eye(2);
H = [0 0];

% Different values of cut-off frequency of noise filter

% Note that the values in the diagonal of the disturbance
% filter inputintensity matrix [1 0; 0 100] are arbitrary;
% their relation will later be varied

A = [-1 0; 0 -0.1];
sys = ss(A, [B N], C, [D H]);
[Kest,L,P] = kalman(sys, [1 0; 0 100], 1);
P = ss(A,B,C,D);
[K,S,E] = lqry(P,1,0.0000001,0);
R1 = lqgreg(Kest,K);

A = [-1 0; 0 -0.001];
```

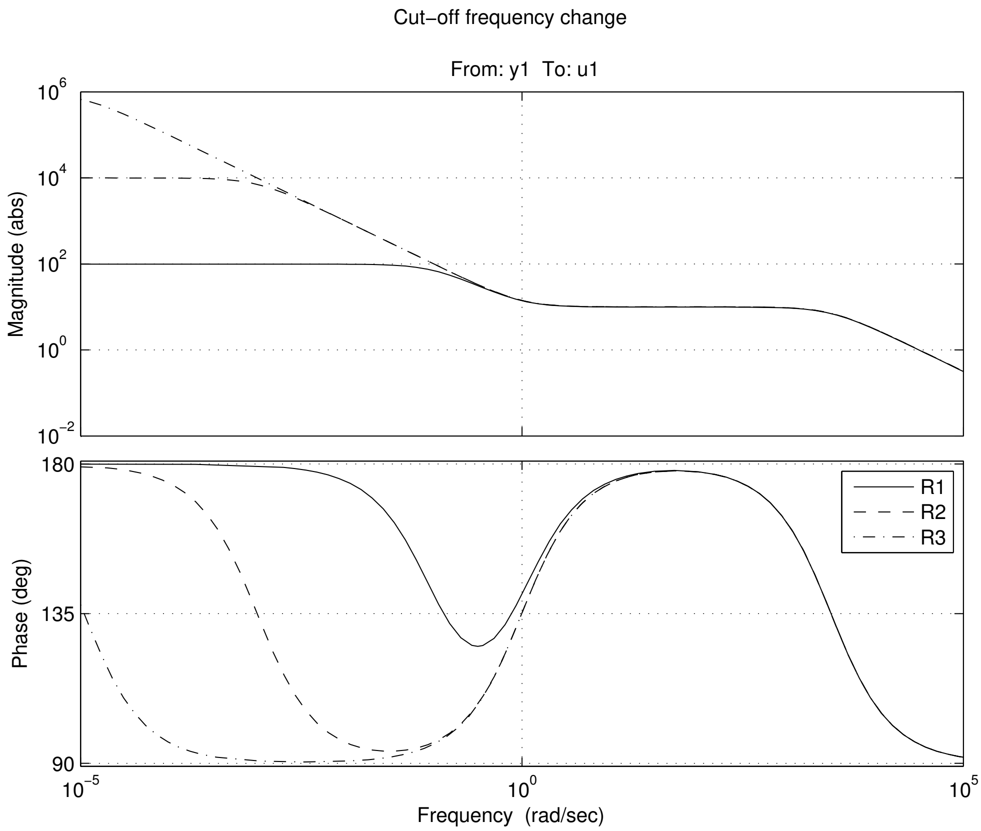

Figure 9.1 Change of cut-off frequency

```
sys = ss(A,[B N],C,[D H]);
[Kest,L,P] = kalman(sys,[1 0; 0 100],1);
P = ss(A,B,C,D);
[K,S,E] = lqry(P,1,0.0000001,0);
R2 = lqgreg(Kest,K);


A = [-1 0; 0 -0.00001];
sys = ss(A,[B N],C,[D H]);
[Kest,L,P] = kalman(sys,[1 0; 0 100],1);
P = ss(A,B,C,D);
[K,S,E] = lqry(P,1,0.0000001,0);
R3 = lqgreg(Kest,K);
figure(1)
bode(R1,R2,'--',R3,'-.'); grid
legend('R1','R2','R3')
title('Cut-off frequency change')

% Different values of the disturbance filter input intensity

A = [-1 0; 0 -0.001];
sys = ss(A,[B N],C,[D H]);
[Kest,L,P] = kalman(sys,[1 0; 0 1],1);
```

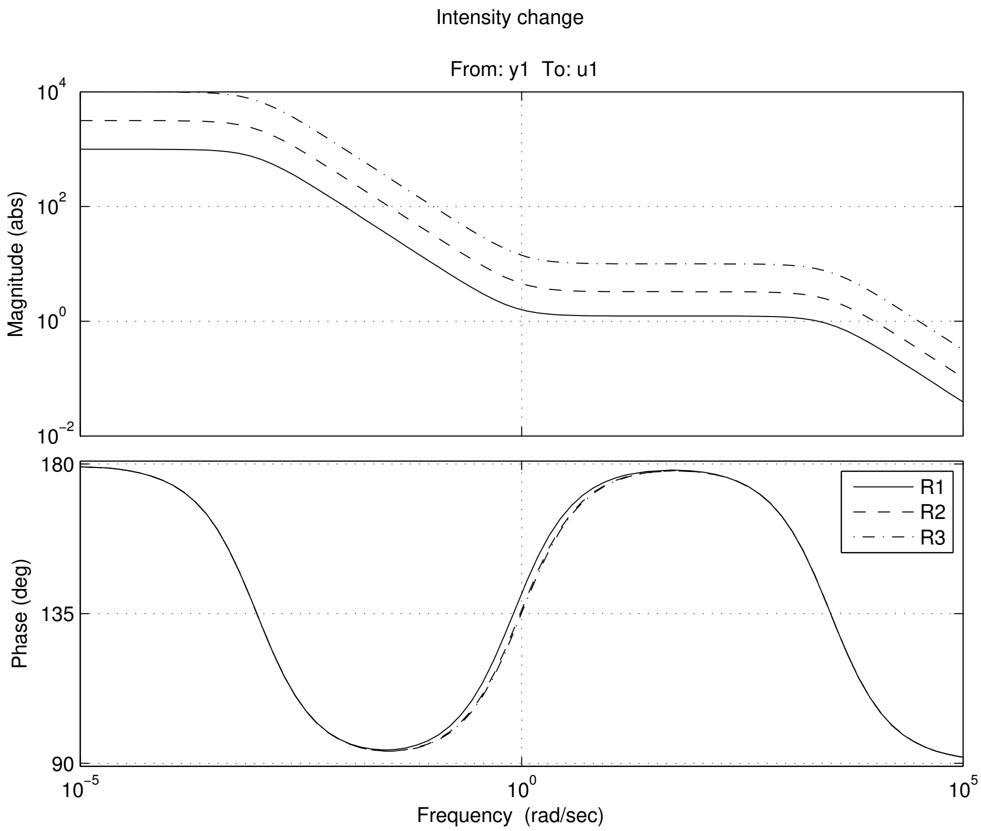

Figure 9.2 Change of noise intensity

```
P = ss(A,B,C,D);
[K,S,E] = lqry(P,1,0.0000001,0);
R1 = lqgreg(Kest,K);

A = [-1 0;0 -0.001];
sys = ss(A, [B N], C, [D H]);
[Kest, L, P]=kalman(sys, [1 0; 0 10], 1);
P = ss(A, B, C, D);
[K,S,E] = lqry(P,1,0.0000001,0);
R2 = lqgreg(Kest,K);

A = [-1 0; 0 -0.001];
sys = ss(A,[B N],C,[D H]);
[Kest,L,P] = kalman(sys,[1 0; 0 100],1);
P = ss(A,B,C,D);
[K,S,E] = lqry(P,1,0.0000001,0);
R3 = lqgreg(Kest,K);

figure(2)
bode(R1,R2,'--',R3,'-.');grid
legend('R1','R2','R3')
title('Intensity change')
```

9.4 **a.** To get a small complementary sensitivity at the oscillation frequency, we need the LQG controller to have a low gain at this frequency; effectively ignoring corresponding oscillations in the output y. This can be achieved by modelling the influence of the oscillatory system as a disturbance w on y according to

$$\begin{aligned} \dot{x} &= Ax + Bu + Nv_1 \\ y &= Cx + w + v_2 \end{aligned}$$

To model the oscillatory characteristics of w, we can consider w to be generated by passing white noise n through a second-order filter with a resonance peak at $\omega_0 = 0.5$ rad/s and a zero at $s = 0$, with transfer function

$$H(s) = \frac{K_v s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}.$$

The zero at $s = 0$ is placed there to avoid an increased gain at low frequencies, which would otherwise follow. It is not necessary unless it is important to avoid this phenomenon and the exercise can be solved without it, which will then yield a slightly different solution to the one below.

The parameter ζ determines the magnitude of the resonance peak, and we can choose e.g. $\zeta = 0.02$.

In state-space form, the filter is given by

$$\begin{aligned} \dot{x}_v(t) &= \begin{pmatrix} -0.02 & -0.25 \\ 1 & 0 \end{pmatrix} x_v(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} n(t) \\ w(t) &= \begin{pmatrix} K_v & 0 \end{pmatrix} x_v(t) \end{aligned}$$

Extend the original state-space form with the noise model

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -0.02 & -0.25 \\ 0 & 0 & 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 0.1 & 0 \\ 0.1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1(t) \\ n(t) \end{pmatrix}$$
$$y(t) = \begin{pmatrix} 1 & 0 & K_v & 0 \end{pmatrix} x(t) + v_2(t)$$
$$z(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} x(t)$$

If this model is used to compute K in the Kalman filter, for an appropriate value of K_v, we get supression of the resonance frequency. The intensity of the added noise input can e.g. be set to 1 since we can control the amplitude of the disturbance by changing K_v. Thus, we have the intensity matrices $R_1 = \text{diag}(1,\ 1)$, $R_2 = 0.1$.

Note that $z(t)$ do not depend on the x_v-states, i.e., if we are about to design an LQG controller, we have no weight on the added noise. The added noise is only used for specifying at what frequencies our measurements are uncertain.

b. See figure 9.3 for the Bode plot of the transfer function from measurement $y(t)$ to estimated output $\hat{y}(t)$ using $K_v = 1$. We see a large attenuation of frequencies at $\omega = 0.5$ rad/s.

Matlab code

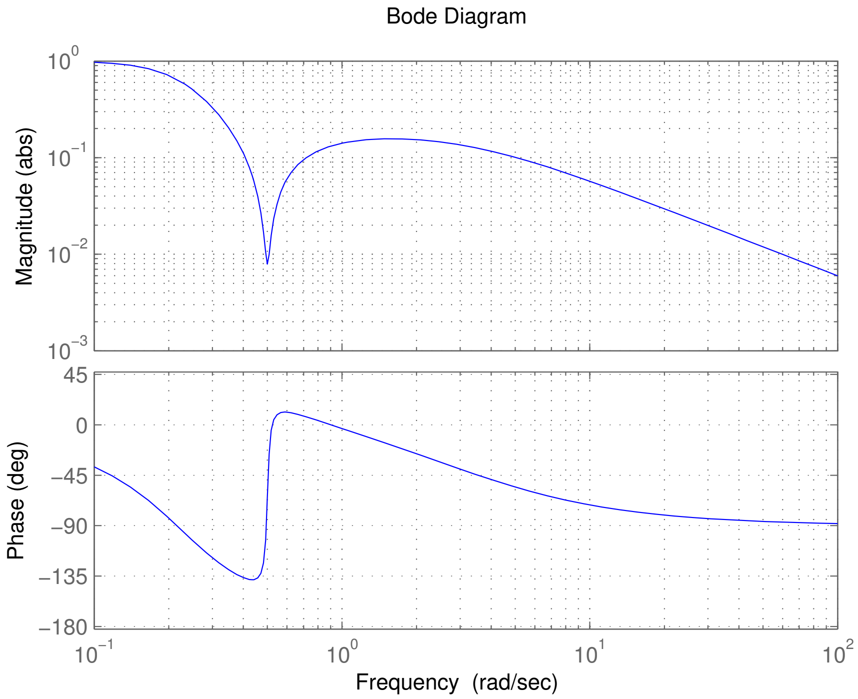

Figure 9.3 Attenuation of oscillative disturbance

```
A = [0 1 0 0; 0 -1 0 0; 0 0 -0.02 -0.2501; 0 0 1 0];
B = [0 1 0 0]'
C = [1 0 1 0];
N = [0.1 0; 0.1 0; 0 1; 0 0];

[K,P,E] = lqe(A,N,C,blkdiag(1,1),0.1);

Cnom = [1 0 0 0];
tf(ss(A-K*C,K,Cnom,0))
bode(ss(A-K*C,K,Cnom,0),{0.1,100})
grid
```

9.5 a. We have the state-space representation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} v_1(t)$$
$$y(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} + v_2(t)$$

(If a different state-space representation is chosen, the solution will look different although the steps will be similar.)

The Riccati-equation

$$AP + PA^T + NR_1N^T - PC^TR_2^{-1}CP = 0$$

is solved by letting $P = \begin{pmatrix} p_1 & p_2 \\ p_2 & p_3 \end{pmatrix}$. The equations become,

$$
\begin{aligned}
2p_2 - p_1^2 &= 0 \\
p_3 - p_1 p_2 &= 0 \\
1 - p_2^2 &= 0
\end{aligned}
$$

The solution is thus

$$P = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$$

with the optimal gain

$$K = PC^T = \begin{pmatrix} \sqrt{2} & 1 \end{pmatrix}^T$$

b. The poles of the Kalman filter are the eigenvalues of $A - KC$,

$$A - KC = \begin{pmatrix} -\sqrt{2} & 1 \\ -1 & 0 \end{pmatrix}$$

with the eigenvalues $\lambda_j = \dfrac{1}{\sqrt{2}}(-1 \pm i)$.