FRTN10 Exercise 8. Linear-Quadratic Control

8.1 Consider the first-order unstable process

$$\begin{aligned}\dot{x}(t) &= ax(t) + u(t), \qquad a > 0 \\ y(t) &= x(t)\end{aligned}$$

where the state is measured without any noise.

a. Design, analytically, an LQ controller that minimizes the criterion

$$J = \int_0^\infty \left(x^2(t) + Ru^2(t)\right) dt.$$

We want a stationary gain of 1 from the reference to the output. Design therefore a feedforward gain L_r such that the control signal is given by

$$u(t) = -Lx(t) + L_r r(t),$$

and achieves the performance specification.

b. Do the design for different R using Matlab when assuming $a = 1$, and plot the position of the closed-loop pole as a function of R. Explain how the speed of the system depends on R.

8.2 Consider the second-order system

$$\begin{aligned}\dot{x}(t) &= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) \\ y(t) &= \begin{pmatrix} 1 & 1 \end{pmatrix} x(t)\end{aligned}$$

Design an LQ controller, with equal weight on output and control signal, by

1. Using `lqry` in Matlab. Simulate the closed-loop system from the initial condition $x(0) = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$.
2. Solving the algebraic Riccati equation in Matlab using `care`.

8.3 Consider a process

$$\dot{x}(t) = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix} x(t) + \begin{pmatrix} 3 \\ 2 \end{pmatrix} u(t)$$

Show that

$$u(t) = -\begin{pmatrix} 2 & -3 \end{pmatrix} x(t)$$

can *not* be an optimal state feedback designed using linear quadratic theory with the cost function

$$J = \int_0^\infty \left(x^T(t) Q_1 x(t) + Q_2 u^2(t)\right) dt$$

where Q_1 and Q_2 are positive definite matrices.

Hint: Look at the Nyquist plot of the loop transfer function.

8.4 Consider the system

$$\dot{x} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix} x + \begin{pmatrix} -4 \\ 8 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 1 \end{pmatrix} x$$

One wishes to minimize the criterion

$$V(T) = \int_0^T \left(x^T(t) Q_1 x(t) + Q_2 u^2(t) \right) dt$$

Is it possible to find positive definite weights Q_1 and Q_2 such that the cost function $V(T) < \infty$ as $T \to \infty$?

8.5 We would like to control the following process with linear quadratic optimal control:

$$\dot{x}(t) = \begin{pmatrix} 1 & 3 \\ 4 & 8 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0.1 \end{pmatrix} u(t)$$
$$z(t) = \begin{pmatrix} 0 & 1 \end{pmatrix} x(t)$$

The weight on $x_1(t)^2$ should be 1, and the weight on $x_2(t)^2$ should be 2. On the control signal $u(t)^2$ we will try different values: $R = 0.01,\ 10,\ 1000$.

a. Determine the cost function for the three different cases.

b. Assume that we want to add reference following, i.e. $u(t) = L_r r(t) - Lx(t)$. In Matlab, calculate the three different resulting controllers, calculate the resulting closed-loop poles and do step responses from r to x_2 and from r to u. Make sure that there is no static error!

8.6 Consider the double integrator

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$
$$z(t) = \begin{pmatrix} 1 & 0 \end{pmatrix} x(t)$$

a. Design by hand an LQ controller $u(t) = -Lx(t)$ that minimizes the criterion

$$J = \int_0^\infty x^T(t) Q_1 x(t) + Q_2 u^2(t) dt$$

with

$$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 0.1$$

Then add reference following so that $u(t) = -Lx(t) + L_r r(t)$. Calculate L_r so that the stationary gain from reference $r(t)$ to output $z(t)$ is equal to 1.

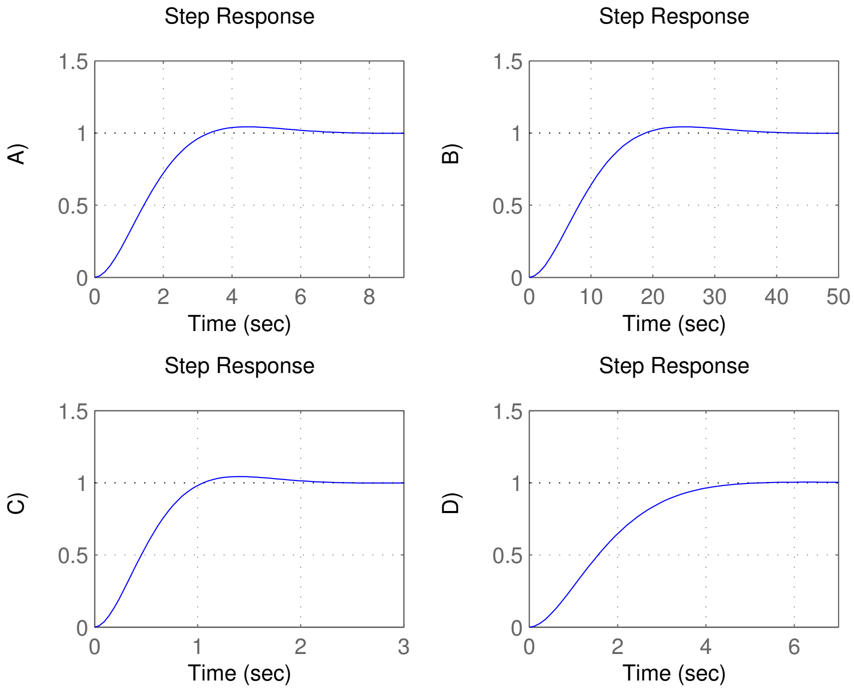

Figure 8.1 Step responses for LQ-control of the system in Problem 8.6 with different weights on Q_1, Q_2.

b. What measurements are needed by the controller?

c. The four plots in Figure 8.1 show the step responses of the closed-loop system for four different combinations of weights, Q_1, Q_2. Pair the combinations of weights given below with the step responses in Figure 8.1.

1. $$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 0.01$$

2. $$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1$$

3. $$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q_2 = 1$$

4. $$Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Q_2 = 1000$$

8.7 (*) Consider the double integrator

$$\ddot{\xi}(t) = u(t).$$

with state-space representation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} x$$

where $x = (\xi(t), \dot{\xi}(t))$. You would like to design a controller using the criterion

$$\int_0^\infty (\xi^2(t) + \eta \cdot u^2(t))\, dt$$

for some $\eta > 0$.

a. Show that $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$ with

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

b. What are the closed-loop poles of the system when using this optimal state feedback? What happens with the control signal if η is reduced?

Solutions to Exercise 8. Linear-Quadratic Control

8.1 **a.** The Riccati equation becomes ($A = a,\ B = 1,\ M = 1,\ Q_1 = 1,\ Q_2 = R$)

$$2Sa + 1 - SR^{-1}S = 0$$

This gives

$$S = aR + \sqrt{(aR)^2 + R}$$

($S = aR - \sqrt{(aR)^2 + R}$ is not a solution since S has to be positive definite.) Thus the optimal control is given by

$$L = \frac{S}{R} = a + \sqrt{a^2 + \frac{1}{R}}.$$

The closed-loop system is hence, using $u(t) = -Lx(t) + L_r r(t)$

$$\begin{aligned}\dot{x}(t) &= -\sqrt{a^2 + \frac{1}{R}}x(t) + L_r r(t)\\ y(t) &= x(t)\end{aligned}$$

L_r has to be chosen so that we get a stationary gain of 1 from the reference to the output, i.e. $G_{r\to y}(0) = C(-A + BL)^{-1}BL_r + D = 1$.
We get $L_r = (L - a) = \sqrt{a^2 + \frac{1}{R}}$.

b. See Matlab code below and Figure 8.1. Conclusion: Less weight on u gives a faster system since we are allowed to move the control signal more, and vice versa.

```
A = 1;
B = 1;
C = 1;

P = ss(A,B,C,0);

Q = 1;
Rvec = 0.001:0.001:0.5;
Evec = zeros(size(Rvec));

for i = 1:length(Rvec)
    R = Rvec(i);
    [L,S,E] = lqr(P,Q,R);
    Evec(i) = E;
end

plot(Rvec, Evec)
xlabel('Control signal weight')
ylabel('Closed-loop pole')
grid
```

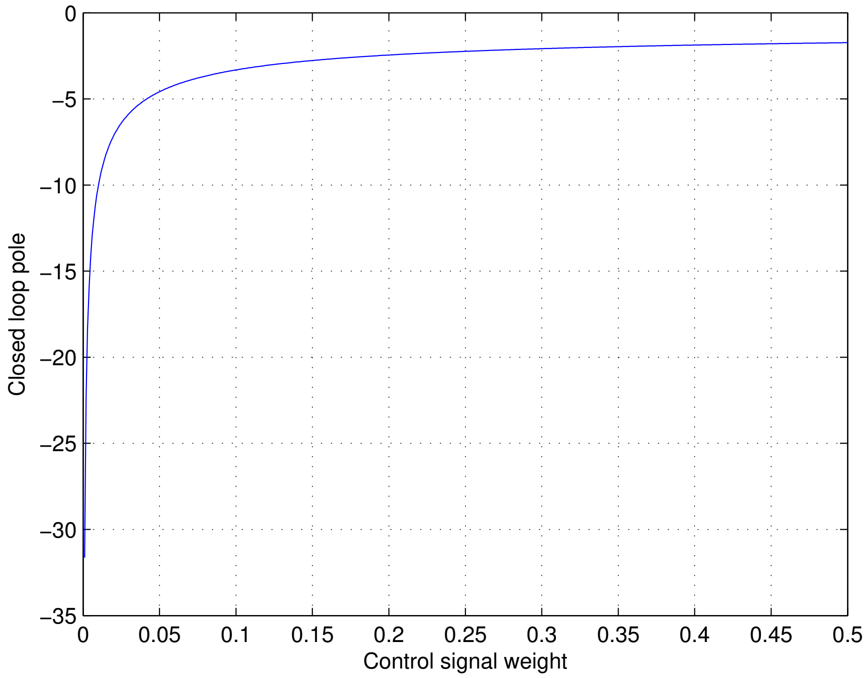

Figure 8.1 Control signal weight versus closed-loop pole

8.2 See Figure 8.2 and Matlab code below

```
A = [1 0; 1 0];
B = [1 0]';
C = [1 1];
Q = 1;
R = 1;

% using lqry
sys = ss(A,B,C,0);
[L2,S,E] = lqry(sys,Q,R)
eig(A-B*L2)

% simulate the system with initial conditions
sys = ss(A-B*L2,B,C,0);
x0 = [1 1];
initial(sys,x0); grid

% Solving the Riccati equation
Qr = C'*Q*C;
Rr = R;
S = zeros(2,1);
E = eye(2);

[X,K,G] = care(A,B,Qr,Rr,S,E);
L1 = Rr\B'*X
eig(A-B*L1)
```

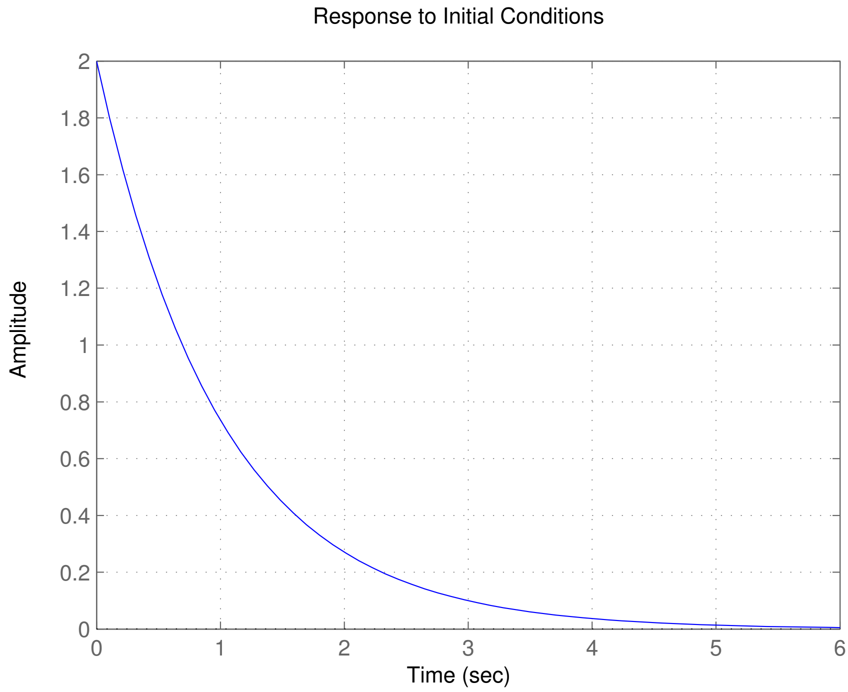

Figure 8.2 Response to initial conditions

8.3 The loop gain is

$$L(sI - A)^{-1}B = \frac{6}{(s+1)(s+2)}$$

Thus, as seen in figure 8.3, the Nyquist curve will approach the origin with a phase of $-180°$. LQ-optimal loop gain always has an asympototic phase of $-90°$. Therefore, it can not be an LQ-optimal state feed back vector.

8.4 The system has two unstable poles in 2 and 3. If the cost function should be less than ∞ then the system must be stabilizable, i.e. all unstable poles must be controllable (due to $Q_1 > 0$). The controllability matrix is given by

$$W_c = (B \quad AB) = \begin{pmatrix} -4 & -12 \\ 8 & 24 \end{pmatrix}$$

which is a rank 1 matrix. Thus, only one of the modes is controllable meaning that there is an uncontrollable, unstable mode, and hence, we can not make the cost function less than ∞.

8.5 **a.** The cost function is $J = \int_0^\infty x^T(t) \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x(t) + u^T(t) R u(t) dt$, $R = 0.01,\ 10,\ 1000.$

b. See Figure 8.4 for step responses, and Matlab code below.

```
A = [1 3; 4 8]; B = [1; 0.1]; M = [0 1];
P = ss(A,B,M,0);
```

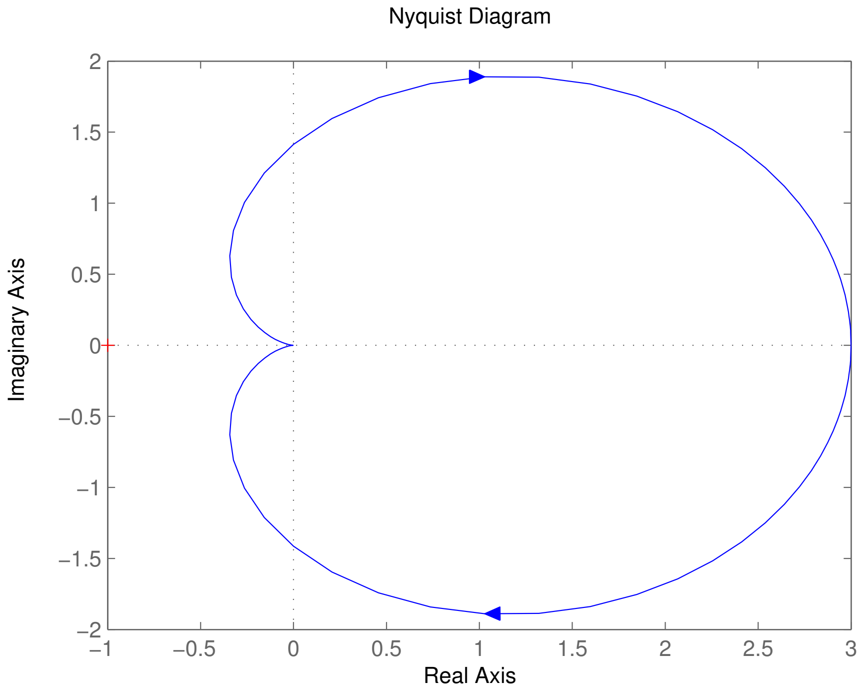

Figure 8.3 Nyquist plot.

```
Q1 = [1 0; 0 2]; Q2_vector = [0.01 10 100];

for i=1:length(Q2_vector)
   [L,S,E] = lqr(P,Q1,Q2_vector(i));

   % Calculating Lr (static gain to output must be 1)
   Lr = 1/(M/(B*L-A)*B);

   % Calculating the control signal:
   to_control_signal = Lr-L*ss(A-B*L,B*Lr,eye(2),0);

   % Calculating the output signal:
   to_output_signal = ss(A-B*L,B*Lr,M,0);

   % Plotting step responses
   figure(11)
   subplot(3,2,i*2-1)
   step(to_control_signal)
   axis([0 10 -Inf Inf])
   title(['Control signal, Q_2=' num2str(Q2_vector(i))])
   subplot(3,2,i*2)
   step(to_output_signal)
   axis([0 10 -Inf Inf])
   title(['Output signal, Q_2=' num2str(Q2_vector(i))])
   poles{i} = E;
end
poles{:}
```

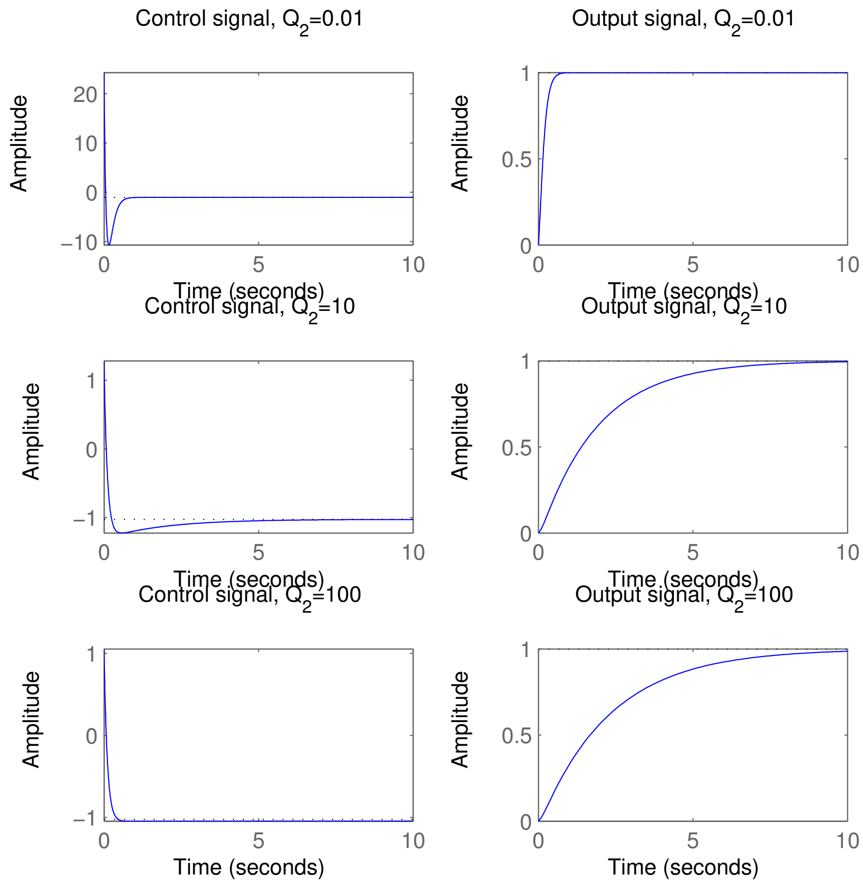

Figure 8.4 Step responses for different weight on control signal.

8.6 **a.** Put

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$$

and solve the Ricatti equation

$$Q_1 + A^T S + SA - SBQ_2^{-1}B^T S = 0.$$

This gives

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{0.1} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0,$$

with the solution

$$\begin{aligned} s_1 &= \sqrt{2} \cdot 10^{-1/4}, \\ s_2 &= 10^{-1/2}, \\ s_3 &= \sqrt{2} \cdot 10^{-3/4}. \end{aligned}$$

The optimal controller is given by

$$L = Q_2^{-1} B^T S = (\sqrt{10} \quad \sqrt{2} \cdot 10^{1/4}).$$

To get $y = r$ in stationarity:

$$1 = G(0) = M(-A + BL)^{-1} BL_r \quad \Rightarrow L_r = \sqrt{10}.$$

b. Both x_1 and x_2 must be measured, e.g.

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

c. 3. is the only case with a cost on the velocity x_2. This makes the controller try to avoid rapid variations in x_1, so we get $3. - D)$, the only step response without overshoot. The weight, Q_2, on the control signal determines the speed of the system. A low weight on the control signal gives a faster system since we are allowed to use more control signal. This gives $1. - C)$, $2. - A)$, $4. - B)$.

8.7 a. Weighting matrices $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ och $Q_2 = \eta$. The Riccati equation to be solved with respect to S is

$$A^T S + SA + Q_1 - SBQ_2^{-1} B^T S = 0$$

Put

$$S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix},$$

which gives

$$\begin{pmatrix} 0 & 0 \\ s_1 & s_2 \end{pmatrix} + \begin{pmatrix} 0 & s_1 \\ 0 & s_2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \frac{1}{\eta} \cdot \begin{pmatrix} s_2^2 & s_2 s_3 \\ s_2 s_3 & s_3^2 \end{pmatrix} = 0$$

We see, by insertion, that

$$\begin{aligned} s_1 &= \sqrt{2} \cdot \eta^{1/4} \\ s_2 &= \eta^{1/2} \\ s_3 &= \sqrt{2} \cdot \eta^{3/4} \end{aligned}$$

solves the Riccati equation.

b. The optimal state feedback is

$$\begin{aligned} L = Q_2^{-1} B^T S &= \frac{1}{\eta} \cdot \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2}\eta^{1/4} & \eta^{1/2} \\ \eta^{1/2} & \sqrt{2} \cdot \eta^{-3/4} \end{pmatrix} \\ &= \frac{1}{\eta} \cdot \begin{pmatrix} \eta^{1/2} & \sqrt{2}\eta^{3/4} \end{pmatrix} = \begin{pmatrix} \eta^{-1/2} & \sqrt{2} \cdot \eta^{-1/4} \end{pmatrix} \end{aligned}$$

The poles are the eigenvalues to $A - BL$. Put $\mu = \eta^{-1/4} \Rightarrow L = \begin{pmatrix} \mu^2 & \sqrt{2} \cdot \mu \end{pmatrix}$. This gives

$$0 = \det \begin{pmatrix} s & -1 \\ \mu^2 & s + \sqrt{2} \cdot \mu \end{pmatrix} = s^2 + \sqrt{2}\mu s + \mu^2,$$

that is

$$s = -\frac{\mu}{\sqrt{2}} \pm \sqrt{\frac{\mu^2}{2} - \mu^2} = -\frac{\mu}{\sqrt{2}} \pm i \cdot \frac{\mu}{\sqrt{2}} =$$

$$= -\frac{\mu}{\sqrt{2}} \cdot (1 \pm i) = -\frac{1}{\sqrt{2} \cdot \eta^{1/4}} \cdot (1 \pm i)$$

If η is reduced, the distance between the poles and the origin will increase. This means that $u(t)$ will increase. Check the criterion!