

FRTN10 Exercise 3. Disturbance Models and Robustness

- 3.1 a. Analyze the stability of the system to the left in Figure 3.1 using the small gain theorem.
- b. Analyze how the stability of the system to the right in Figure 3.1 depends on the constant feedback gain K . If you get different results from a, explain why!

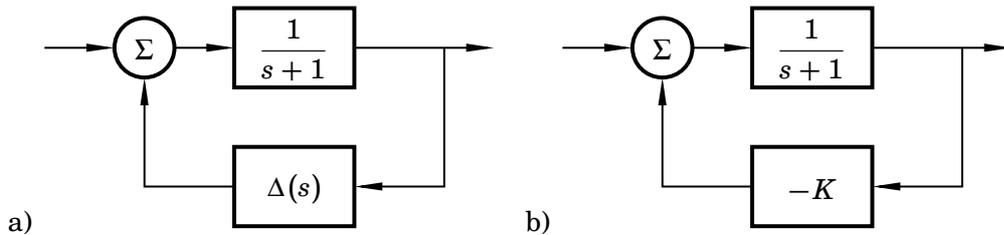


Figure 3.1

- 3.2 Consider the system in Figure 3.2.

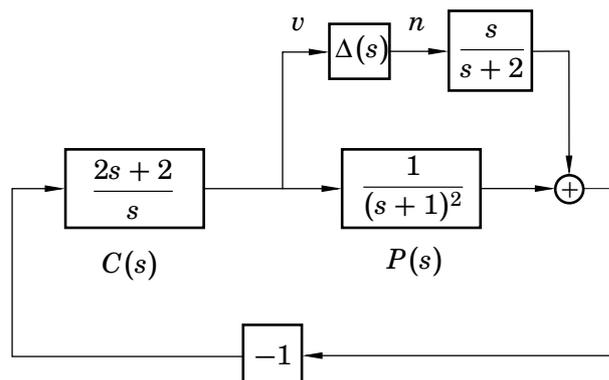


Figure 3.2 System for Problem 3.2.

- a. Find the transfer function G_{vn} from n to v .
- b. How large is the gain $\|G_{vn}\|_{\infty}$? Support the solution by a Matlab plot.
- c. Using the small gain theorem, find the largest possible L_2 -gain of Δ for which the closed-loop system is stable.
- d. The Δ block is used to account for uncertainty in the process model. Explain the role of the factor $\frac{s}{s+2}$ multiplying Δ .
- 3.3 A feedback system is shown in Figure 3.3.
- a. Compute the poles of the closed-loop system.

Exercise 3. Disturbance Models and Robustness

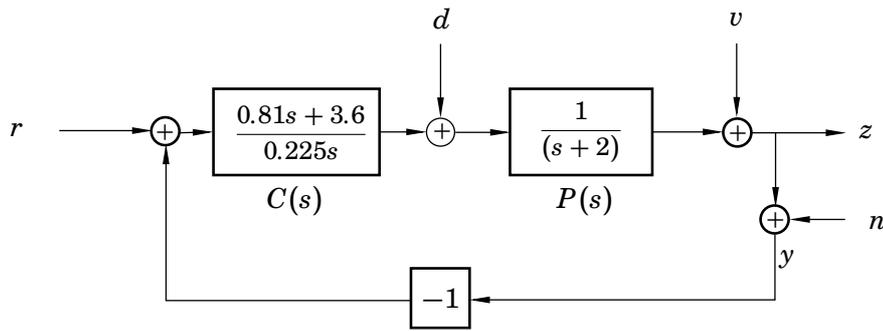


Figure 3.3 System in Problem 3.3.

- b. Derive the transfer functions from r, d, v and n to z . Identify the sensitivity function S and the complementary sensitivity function T . Plot them in the same Bode plot.
- c. If we have a disturbance $v = \sin(0.5t)$ acting on the system and all other input signals are zero, what amplitude would the oscillations in the output signal have when the transients have disappeared?
- d. If we have sinusoidal measurement disturbances $n(t)$ with a frequency of 50 Hz and unit amplitude and all other input signals are zero, what amplitude would the oscillations in the output signal have when the transients have disappeared?

3.4 (*) Consider the system in Figure 3.4.

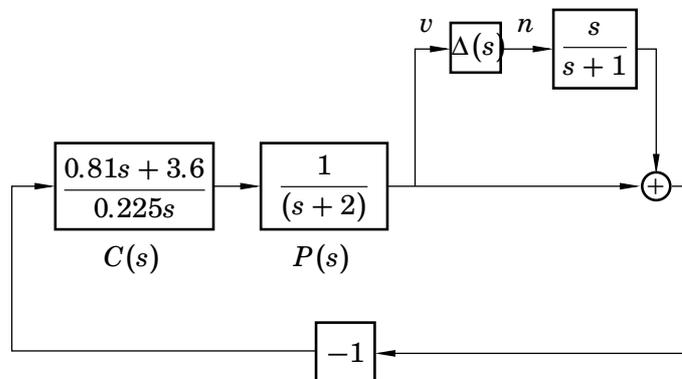


Figure 3.4 System for Problem 3.4.

- a. What is the largest possible bound on the L_2 -gain of Δ , for which the closed-loop system is stable, by the small gain theorem?
- b. If $\Delta(s)$ is replaced by a real parameter δ , then for what values of δ is the closed loop stable? Compare with the gain bound in (a).
- c. What is the difference between the uncertainty model in this problem and the one in Figure 3.2?

- 3.5** A continuous-time stochastic process $u(t)$ has the power spectrum $\Phi_u(\omega)$. The process can be represented by a linear filter that has white noise as input. Determine the linear filter when

a.

$$\Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2}$$

b.

$$\Phi_u(\omega) = \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}$$

- 3.6** Consider a missile travelling in the air. It is propelled forward by a jet force u along a horizontal path. The coordinate along the path is z . We assume that there is no gravitational force. The aerodynamic friction force is described by a simple model as

$$f = k_1 \cdot \dot{z} + v,$$

where v are random variations due to wind and pressure changes. Combining this with Newton's second law, $m\ddot{z} = u - f$, where m is the mass of the missile, gives the input-output relation

$$\ddot{z} + \frac{k_1}{m}\dot{z} = \frac{1}{m}(u - v).$$

- a. Express the input-output relation in state-space form.
 b. The disturbance v has been determined to have the spectral density

$$\Phi_v(\omega) = k_0 \cdot \frac{1}{\omega^2 + a^2}$$

Expand your state-space description so that the disturbance input can be expressed as white noise.

- 3.7 (*)** This problem builds on Problem 3.6.

- a. Assume that the position measurement is distorted by an additive error $n(t)$,

$$y(t) = z(t) + n(t)$$

Write down the state-space equations for the system, assuming that $n(t)$ is white noise with intensity 0.1, i.e. $\Phi_n(\omega) \equiv 0.1$.

- b. Solve the same problem, this time with

$$\Phi_n(\omega) = 0.1 \frac{\omega^2}{\omega^2 + b^2}.$$

- c. Solve the problem with

$$\Phi_n(\omega) = 0.1 \frac{1}{\omega^2 + b^2}.$$

Exercise 3. Disturbance Models and Robustness

3.8 (*) Consider a swing that is hanging outside in the wind. The swing is described by the transfer function

$$Y(s) = \frac{1}{s^2 + s + 1}U(s)$$

where the output signal $y(t)$ is the angle of the swing (relative to the vertical axis) and the input signal $u(t)$ is the moment around the pivotal point. The influence of wind can be described as

$$u(t) = Kv(t)$$

where $v(t)$ is a normally distributed disturbance with spectrum

$$\Phi_v(\omega) = \frac{2\alpha}{\alpha^2 + \omega^2}, \quad \alpha > 0.$$

K is a measure of the wind strength and α is a measure of the occurrence of wind gusts.

a. Does α increase or decrease when there are more wind gusts (i.e., when the wind changes strength and direction more often)?

b. Determine the variance of $y(t)$. What is your interpretation?

Hint: Equation (5.25) in Glad & Ljung might be helpful, also

$$\begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{|b_2(i\omega)^2 + b_1i\omega + b_0|^2}{|(i\omega)^3 + a_2(i\omega)^2 + a_1i\omega + a_0|^2} d\omega \\ = \frac{b_2^2 a_0 a_1 + (b_1^2 - 2b_0 b_2) a_0 + b_0^2 a_2}{2a_0(-a_0 + a_1 a_2)} \end{aligned}$$

3.9 (*) Consider an electric motor with the transfer function

$$G(s) = \frac{1}{s(s+1)}$$

from input current to output angle.

There are two different disturbance scenarios:

(i) $Y(s) = G(s)(U(s) + W(s))$

(ii) $Y(s) = G(s)U(s) + W(s)$

In both cases, $w(t) = v(t)$, where $v(t)$ is a unit disturbance, e.g., an impulse.

a. Draw a block diagram of the two cases.

b. Put both cases in state-space form. It is assumed in the second case that the disturbance does not give cause to any common states with the engine.

c. Give a physical interpretation of $w(t)$ is, in the two cases.

Solutions to Exercise 3. Disturbance Models and Robustness

3.1 Let

$$G(s) = \frac{1}{s+1}.$$

- a.** The closed loop is guaranteed to be stable according to the small gain theorem if $\|G\| \cdot \|\Delta\| < 1$. We see that $\|G\| = 1$ (use $\text{norm}(G, \text{inf})$ or look at the Bode Plot), so the system is guaranteed to be stable if $\|\Delta\| < 1$.
- b.** The loop gain is given by

$$L(s) = G(s)K = \frac{K}{s+1},$$

and the sensitivity function by

$$S = \frac{1}{1+L} = \frac{1}{1 + \frac{K}{s+1}} = \frac{s+1}{s+1+K}.$$

We see that there is one closed-loop pole in $s = -(1+K)$, so the system is stable exactly when $K > -1$. We can compare this to the result in **a**, which guarantees that the system is stable when $|K| < 1$.

The different results arise from the fact that the small gain theorem is conservative in nature, i.e. it gives a *sufficient* condition on stability, but that condition may not be *necessary*. The main reason of such a conservatism, is that there is no *a priori* assumptions on Δ . Δ in **a** can be a transfer function of an arbitrary order, not just an unknown scalar as in **b**. Looking at the closed-loop poles, on the other hand, shows exactly when the system is stable.

3.2 **a.** Block diagrams of the original and the rewritten closed-loop system are shown in Figure 3.1. We have

$$C(s) = \frac{2s+2}{s} \quad P(s) = \frac{1}{(s+1)^2} \quad W(s) = \frac{s}{s+2}$$

$$G_{vn}(s) = -\frac{C(s)W(s)}{1+P(s)C(s)} = -\frac{2s^4 + 6s^3 + 6s^2 + 2s}{s^4 + 4s^3 + 7s^2 + 8s + 4} = -\frac{2s^3 + 4s^2 + 2s}{s^3 + 3s^2 + 4s + 4}$$

Matlab commands:

```
>> s = tf('s');
>> C = 2*(s+1)/s
>> P = 1/(s+1)^2
>> W = s/(s+2);
>> Gvn = -feedback(C,P)*W;
```

- b.** The L_2 -gain of G_{vn} is equal to 2.63. This corresponds to the peak magnitude in the Bode diagram of Figure 3.2.

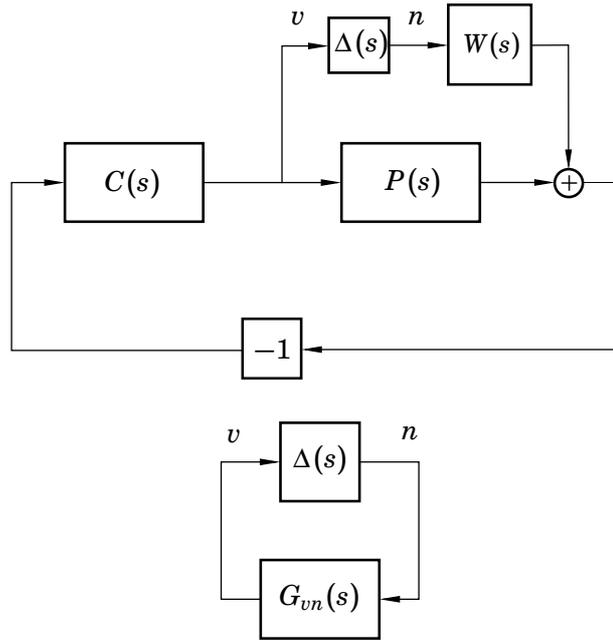


Figure 3.1 Systems for Problem 3.2.

c. The small gain theorem shows stability for all perturbations, Δ , satisfying

$$\|\Delta\|_{\infty} \cdot \|G_{vn}\|_{\infty} < 1$$

The closed-loop system is therefore stable for all perturbations Δ with

$$\|\Delta\|_{\infty} < 1/\|G_{vn}\|_{\infty} = 0.38$$

Matlab commands:

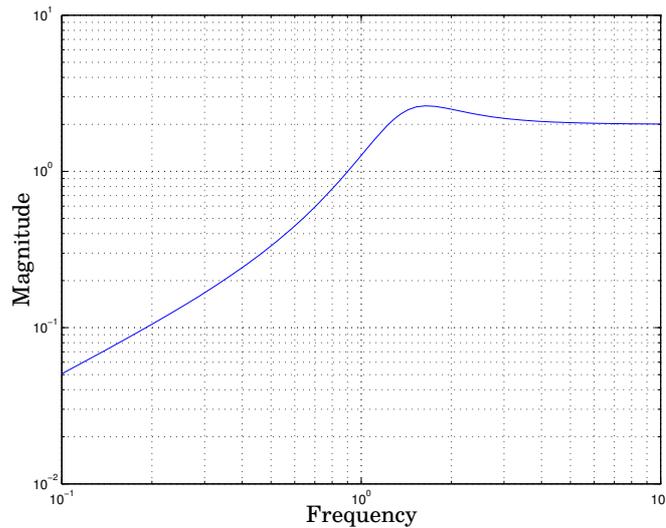


Figure 3.2 Bode magnitude diagram for $G_{vn}(s)$.

```
>> norm(Gvn, inf)
>> 1/ans
```

- d.** Process models are often better (that is, they match the real process more closely) in the low frequency range. As frequency increases there is usually excitation of higher-order dynamics and non-linearities in the real process, which is not covered by the model.

Since we know that there is more uncertainty for high frequencies, this can be used to get some *structure* on the uncertainty block. This structure is given by the extra factor (such factors are usually called *weighting functions*), which effectively makes the uncertainty smaller for low frequencies (approximately when $\omega < 2$). Without this factor, the analysis would assume equal uncertainty for all frequencies, yielding a lower bound on the L_2 -gain of Δ . In other words, the system would appear less robust.

3.3 a. Matlab commands:

```
>> s = tf('s');
>> P = 1/(s+2);
>> C = (0.81*s+3.6)/(0.225*s)
>> G = feedback(C*P,1);
>> pole(G)
```

ans =

```
-2.8000 + 2.8566i
-2.8000 - 2.8566i
```

- b.** The transfer functions are

$$Z = \frac{1}{1+CP}V + \frac{P}{1+CP}D - \frac{CP}{1+CP}N + \frac{CP}{1+CP}R$$

$$Z = SV + SPD + T(R - N)$$

$$Z(s) = \begin{pmatrix} S(s) & S(s)P(s) & T(s) & -T(s) \end{pmatrix} \begin{pmatrix} V(s) \\ D(s) \\ R(s) \\ N(s) \end{pmatrix}$$

Matlab commands:

```
>> T = feedback(C*P,1);
>> S = 1-T;
>> bode(T)
>> hold on
>> bode(S)
>> S
```

Transfer function:

```
      s^2 + 2 s
-----
s^2 + 5.6 s + 16
```

```
>> T
```

Transfer function:

$$3.6 s + 16$$

$$s^2 + 5.6 s + 16$$

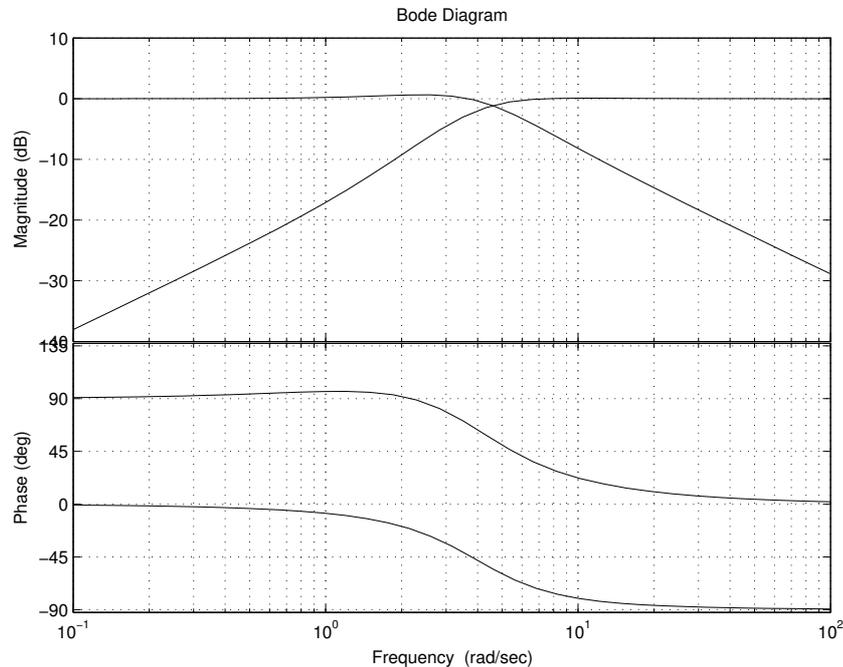


Figure 3.3 Bode diagrams of S and T in Problem 3.3.

- c. In the bode plot of the sensitivity function, we see that $\|S(i0.5)\| = -23.8 \text{ dB} = 10^{(-23.8/20)} = 0.0646$

Matlab commands:

```
>> abs(freqresp(S,0.5))
```

```
ans =
```

```
0.0644
```

- d. We convert $\omega = 2\pi 50 \text{ Hz} = 314.16 \text{ rad/s}$. In the bode plot of the complementary sensitivity function, we see that $\|T(i314.16)\| = -38.8 \text{ dB} = 10^{(-38.8/20)} = 0.0115$

We have very good attenuation of both load disturbances and measurement noise.

- 3.4 a. The transfer function from n to v as seen in Figure 3.4 can be written as $H = \frac{-PCW}{1+PC}$ according to the following Matlab commands:

```
>> s = tf('s');
>> W = s/(s+1);
>> P = 1/(s+2);
>> C = (0.81*s+3.6)/(0.225*s);
>> H = -feedback(P*C,1)*W;
```

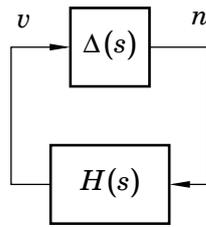


Figure 3.4 Rewritten closed-loop system for Problem 3.4(a).

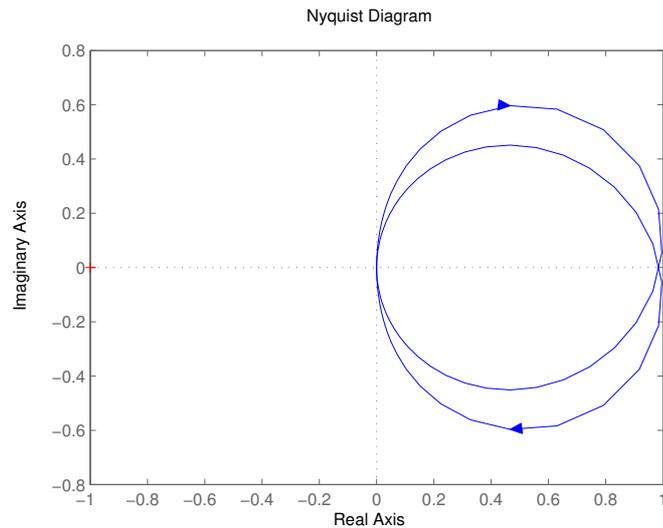


Figure 3.5 Nyquist plot of $-H(s)$ in Problem 3.4(b).

```
>> norm(H,inf)
ans =
    1.0072
>> lower_bound = 1/ans
ans =
    0.9928
```

- b.** We know that $\Delta(s) = \delta$, a real number. Looking at Figure 3.4 we see that we can apply the Nyquist Criterion to analyze the closed-loop stability.

From the Nyquist Plot of $-H(s)$ in Figure 3.5, we see that the closed loop is stable for all $\delta \geq 0$. For negative δ 's, the closed loop will become unstable once the bubble formed by the Nyquist Curve has grown so large that -1 is no longer on its outside. We find this value to be $-\delta = 1.0119$ from the gain margin of $-H$. Thus, the system is stable when $\delta > -1.0119$.

The small gain theorem is easy to use, but it can be conservative, since there is no prior assumptions on structure of uncertainty. With more information about the uncertainty, the bounds can be less conservative and we can allow all positive values of δ as well.

Matlab code:

```
>> nyquist(-H)
```

```
>> allmargin(H)
ans =
    GainMargin: 1.0119
    GMFrequency: 2.3256
    PhaseMargin: [-4.4628 -18.9141]
    PMFrequency: [2.5211 3.1922]
    DelayMargin: [2.4614 1.8649]
    DMFrequency: [2.5211 3.1922]
    Stable: 1
```

- c. In the referenced figure the uncertainty is *added* to the process. This is called an *additive uncertainty*, ie. $P + \Delta$. Here, the uncertainty is multiplied to the output signal, giving a *multiplicative uncertainty*, ie. $P(1 + \Delta)$. In this type of model, the uncertainty is proportional to the process gain.

3.5 $\Phi_u(\omega)$ is an even, scalar, non-negative function. Thus we can divide it into

$$\Phi_u(\omega) = G(i\omega)G(-i\omega)\Phi_e(\omega)$$

where $G(s)$ has its poles and zeroes in the left half-plane and $\Phi_e = 1$ (white noise).

a.

$$\Phi_u(\omega) = \frac{a^2}{\omega^2 + a^2}\Phi_e(\omega) = \frac{a}{i\omega + |a|} \cdot \frac{a}{-i\omega + |a|}$$

So the linear filter is

$$G(s) = \frac{a}{s + |a|}$$

b. In the same way, we get

$$\begin{aligned}\Phi_u(\omega) &= \frac{a^2 b^2}{(\omega^2 + a^2)(\omega^2 + b^2)}\Phi_e(\omega) \\ &= \frac{ab}{(i\omega + |a|)(i\omega + |b|)} \cdot \frac{ab}{(-i\omega + |a|)(-i\omega + |b|)} \\ \Rightarrow G(s) &= \frac{ab}{(s + |a|)(s + |b|)}\end{aligned}$$

3.6 a. To make a state-space description, we let $x_1 = z$, $x_2 = \dot{z} \implies$

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{1}{m}(u - k_1 x_2 - v).\end{aligned}$$

In matrix form:

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{k_1}{m} \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} u + \begin{pmatrix} 0 \\ -\frac{1}{m} \end{pmatrix} v,$$

$$z = (1 \ 0) x.$$

b. We want to find a filter H such that

$$\Phi_v(\omega) = |H(i\omega)|^2 \Phi_e(\omega)$$

Thus $H(s) = \frac{\sqrt{k_0}}{s+|a|}$, which is equivalent to $\dot{v} + |a|v = \sqrt{k_0}e$.

Adding a new state $x_3 = v$ to the state-space description, gives

$$\dot{x}_3 = -|a|x_3 + \sqrt{k_0}e$$

and

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\frac{k_1}{m} & -\frac{1}{m} \\ 0 & 0 & -|a| \end{pmatrix} x + \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 0 \\ \sqrt{k_0} \end{pmatrix} e$$

$$z = (1 \ 0 \ 0)x, \quad \Phi_e(\omega) = 1$$

3.7 a. With $\{A, B, C, N\}$ according to the solution of problem 3.6, we have

$$\begin{aligned} \dot{x} &= Ax + Bu + Ne \\ y &= Cx + n \end{aligned}$$

where n has spectral density $\Phi_n \equiv 0.1$.

b. A noise signal with the specified spectral density is given by the output of a linear system with white noise input that has spectral density $\Phi_{w_n} = 0.1$. The transfer function of the system is

$$G_n(s) = \frac{s}{s+|b|} = \frac{s+|b|-|b|}{s+|b|} = 1 - \frac{|b|}{s+|b|}$$

In state-space form this can be expressed as

$$\begin{aligned} \dot{x}_4 &= -|b|x_4 + |b|w_n \\ n &= -x_4 + w_n \end{aligned}$$

Combining the noise model with our original system gives the expanded state-space description:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A & 0 \\ 0 & -|b| \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & |b| \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix} \\ y &= \begin{pmatrix} C & -1 \end{pmatrix} x + w_n, \quad \Phi_{w_n} = 0.1 \end{aligned}$$

Note that the disturbance can be described using a transfer function and white noise of any spectral density. For instance, it is often convenient to assume white noise with a spectral density of 1. In this case, the transfer function of the system would be

$$G_n(s) = \frac{\sqrt{0.1}s}{s+|b|}$$

The expanded state space description would then need to be adjusted to account for this.

- c. Now, the transfer function of the noise model is $G_n(s) = \frac{1}{s+|b|}$. In state-space form, this is

$$\dot{x}_4 + |b|x_4 = w_n.$$

The expanded system becomes

$$\begin{aligned} \dot{x} &= \begin{pmatrix} A & 0 \\ 0 & -|b| \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e \\ w_n \end{pmatrix} \\ y &= (C \ 1) x, \quad \Phi_{\omega_n} = 0.1 \end{aligned}$$

As in subproblem b, the disturbance can be described using a transfer function and white noise of any spectral density. Assuming white noise with a spectral density of 1, the transfer function of the system would be

$$G_n(s) = \frac{\sqrt{0.1}}{s + |b|}$$

- 3.8 a.** The spectrum of the wind is of low-pass character with cut-off frequency α . When α is increased, $v(t)$ becomes more similar to white noise, i.e. there is more high-frequency content in the signal. Thus, higher α means more wind gusts.

Alternatively, one could look at the covariance function:

$$R_v(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_v(\omega) e^{i\omega\tau} d\omega = e^{-\alpha|\tau|}, \quad \alpha > 0.$$

The covariance function has a sharper peak when α is large. That is, the correlation between $v(t)$ and $v(t + \tau)$ is small, meaning that the wind changes more often.

- b.** Using spectral factorization, the influence of wind can be described as white noise $e(t)$ with intensity 1 filtered through a linear system with transfer function

$$H(s) = \frac{\sqrt{2/\alpha}}{1 + s/\alpha}$$

Thus $Y(s) = G(s) H(s) E(s)$, where

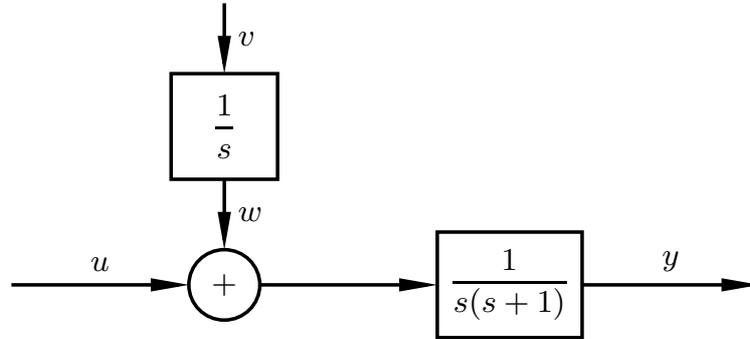
$$G(s) H(s) = \frac{K\sqrt{2\alpha}}{(\alpha + s)(s^2 + s + 1)} = \frac{K\sqrt{2\alpha}}{s^3 + (1 + \alpha)s^2 + (1 + \alpha)s + \alpha}.$$

The variance of the output is

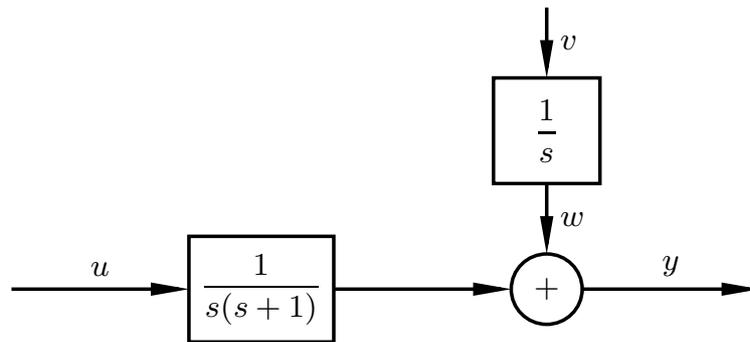
$$\begin{aligned} \text{Var}(y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} |G(i\omega)H(i\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \frac{K\sqrt{2\alpha}}{(i\omega)^3 + (1 + \alpha)(i\omega)^2 + (1 + \alpha)i\omega + \alpha} \right|^2 d\omega \\ &= \frac{K^2(1 + \alpha)}{1 + \alpha + \alpha^2}. \end{aligned}$$

Apparently, the variance increases with wind strength, which is no surprise. However, the variance decreases with the amount of wind gusts. The reason is that a low amount of gusts means that there are longer periods of almost constant wind force, where the swing is displaced far from the origin. A lot of gusts, on the other hand, results in the wind force changing sign frequently, more or less cancelling its own effect a lot of the time.

3.9 a. (i)



(ii)



$v(t)$ is a unit disturbance

b. (i)

$$\dot{x} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}}^A x + \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^B u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$

$$y = \underbrace{(1 \ 0 \ 0)}_C x.$$

(ii)

$$\dot{x} = \overbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}^A x + \overbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}^B u + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v$$

$$y = \underbrace{(1 \ 0 \ 1)}_C x.$$

Solutions 3. Disturbance Models and Robustness

- c.** (i) $w(t)$ could be an offset current on the input to the motor, and/or a step disturbance in the load.
- (ii) In this case $w(t)$ is a measurement disturbance, i.e. an additive error (constant) in the angle measurement. It could also be interpreted as a load disturbance on the process output. A controller could remove the effect from a load disturbance on the process output, but not a constant measurement disturbance, so the interpretation makes a difference.