FRTN10 Exercise 2. System Representations and Stability

2.1 A system is given by

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2 + u_1\\ \dot{x}_2 &= -3x_2 + u_1 + 2u_2\\ y_1 &= x_1 + x_2\\ y_2 &= 2x_1 + u_1\\ y_3 &= 2x_2 + u_2\end{aligned}$$

Express the system in state-space form by determining the matrices (A, B, C, D).

2.2 A system with two inputs and one output is modeled by the differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = b_{11}\dot{u}_1 + b_{12}u_1 + b_{21}\dot{u}_2 + b_{22}u_2.$$

Find the transfer matrix.

2.3 A system has the following input-output relation:

$$y(t) = \int_0^t (t-\tau)e^{-2(t-\tau)}u(\tau)d\tau$$

a. Determine $g(t)$ (the open-loop impulse response) such that

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau$$

Also, if $u(t) = r(t) - y(t)$, find the closed-loop transfer function $G_c(s)$ such that

$$Y(s) = G_c(s)R(s)$$

b. Is the closed-loop system input-output stable?

c. Use Matlab to plot the Bode diagram of the closed-loop system and use that to estimate the L_2 gain of the system.

2.4 In Figure 2.1, a feedback system is illustrated.

a. Determine the transfer function from disturbances v to the controlled output z. This important function, called the *sensitivity function*, is denoted by $S(s)$. (Note: v is sometimes called *process noise* but in some literature also *output load disturbance*).

Also determine the transfer function from reference r to output z. This function, equally important, is called the *complementary sensitivity function*, and is denoted by $T(s)$.

What is the transfer function from measurement noise n to output z, expressed in S and T?

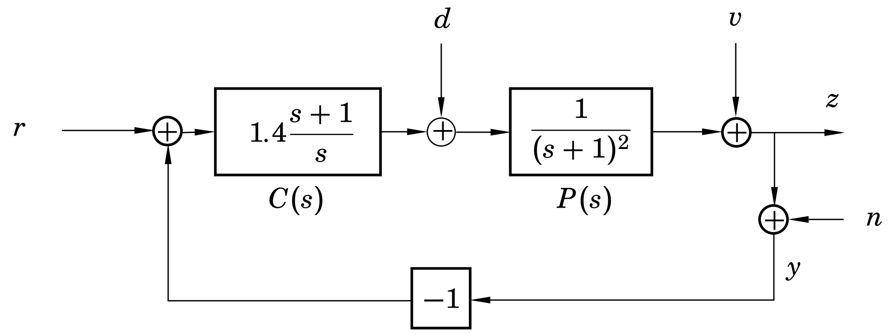

Figure 2.1 System in Problem 2.4

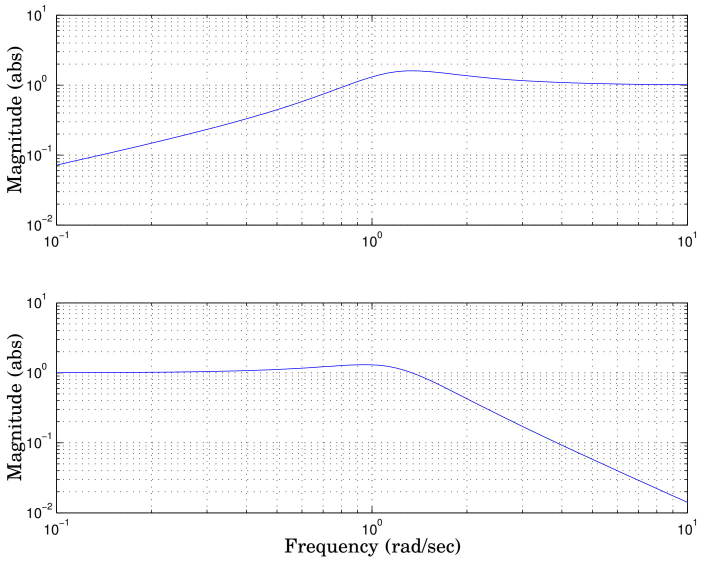

Figure 2.2

b. Figure 2.2 shows gain curves for the sensitivity function and the complementary sensitivity function. Which curve represents which function?

c. In which frequency region (roughly) is there good tracking of the reference value?

d. In which frequency region (roughly) is there good attenuation of the measurement noise, n?

2.5 Study the feedback control system in Figure 2.3, where the process, $P(s)$, is given by

$$P(s) = \frac{1}{(s+1)(s+2)}$$

The Bode diagram of $P(s)$ is shown in Figure 2.4.

Three different controllers were designed

$$C_1(s) = 10 \quad C_2(s) = 10\frac{s+1}{s} \qquad C_3(s) = 10\frac{s+1}{s}e^{-0.1s}$$

where the last one has a small delay.

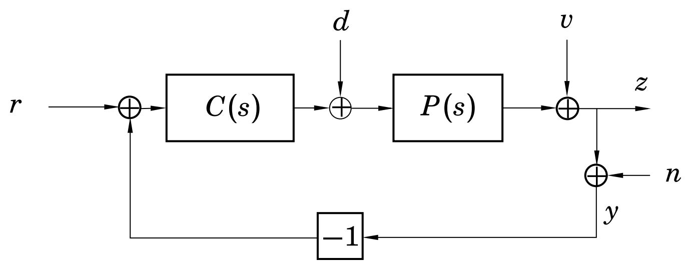

Figure 2.3 System in Problem 2.5

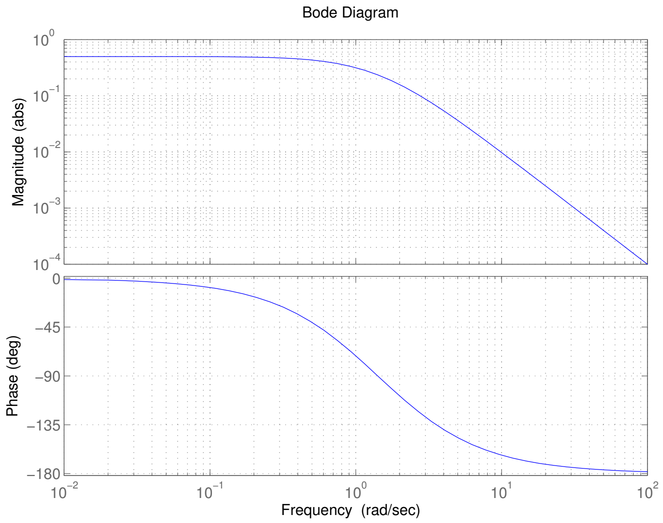

Figure 2.4 Bode diagram for $P(s)$ in Problem 2.5.

a. Figure 2.5 shows sensitivity functions, corresponding to the three different control designs $C_1 - C_3$. Combine the controllers $C_1 - C_3$ with the sensitivity functions $A - C$.

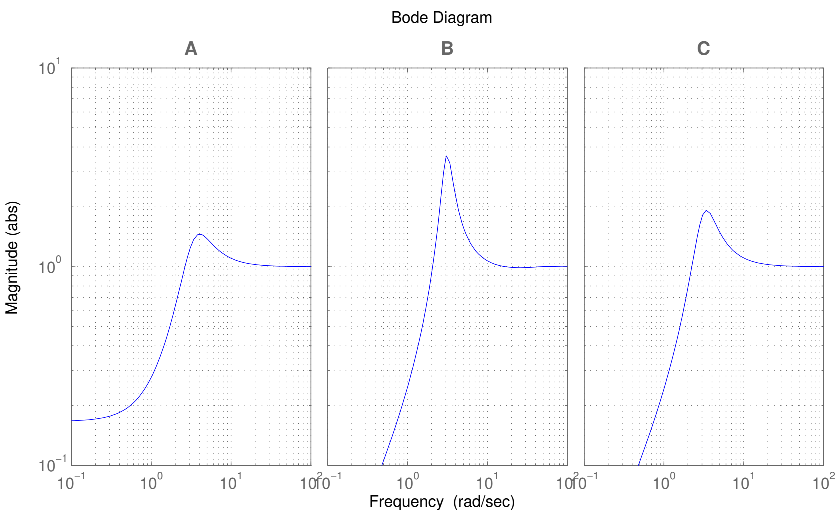

Figure 2.5 Sensitivity functions for Problem 2.5.

b. Figure 2.6 shows responses to a step load disturbance, d, corresponding to the three different control designs $C_1 - C_3$. Combine the controllers $C_1 - C_3$ with the load step responses $I - III$.

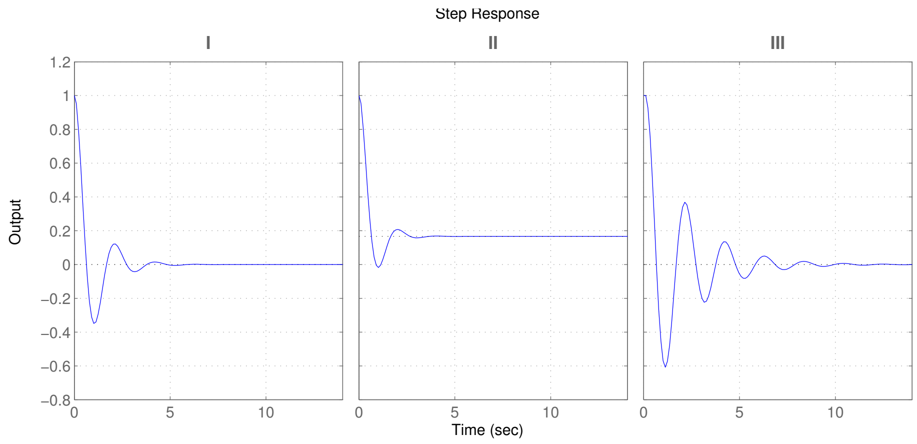

Figure 2.6 Step Load Disturbance Responses for Problem 2.5.

2.6 Consider a water tank with a separating wall. The wall has a hole at the bottom, as can be seen in figure 2.7.

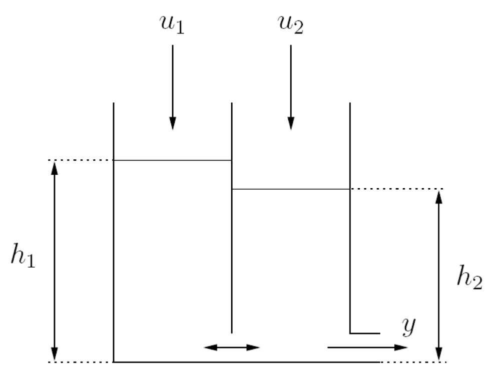

Figure 2.7

The input signals are the inflows of water to the left, u_1, and the right, u_2, halves of the tank, measured in cm^3/s. The water levels are denoted by h_1 cm and h_2 cm, respectively. The outflow y cm^3/s is considered proportional to the water level in the right half of the tank:

$$y(t) = \alpha h_2(t)$$

The flow between the tank halves is proportional to the difference in level:

$$f(t) = \beta(h_1(t) - h_2(t))$$

(flow from left to right)

The signals h_i, u_i and y are thought of as deviations from a linearization point, and may therefore be negative. Assume that the two tank halves each have area $A_1 = A_2 = 1$ cm^2.

a. Write the system in state-space form.

b. What is the transfer matrix from $\begin{pmatrix} u_1 & u_2 \end{pmatrix}^T$ to y?

c. What is the L_2 gain of the system when $\alpha = \beta = 1$? (Hint: Use Matlab.)

d. It turns out that the L_2 gain is larger than one. How is this possible? Can there be more water coming out from the tank than what is poured into it? Have we invented a water-producing device? Explain what is wrong here!

Solutions to Exercise 2. System Representations and Stability

2.1 A state-space representation of the system is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

2.2 Laplace transformation of the differential equation gives

$$Y(s) = \frac{(b_{11}s + b_{12})}{(s^2 + a_1 s + a_2)} U_1(s) + \frac{(b_{21}s + b_{22})}{(s^2 + a_1 s + a_2)} U_2(s)$$

The transfer matrix becomes

$$\begin{pmatrix} \frac{b_{11}s+b_{12}}{s^2+a_1 s+a_2} & \frac{b_{21}s+b_{22}}{s^2+a_1 s+a_2} \end{pmatrix}$$

2.3 **a.** The equation can be written as

$$y = g * u \tag{2.1}$$

where $g(t) = te^{-2t}$, $t \geq 0$. Taking the Laplace transform of (2.1) gives with $u = r - y$

$$Y(s) = \frac{1}{(s+2)^2}(R(s) - Y(s))$$
$$Y(s) = \frac{1}{s^2 + 4s + 5} R(s)$$

b. The transfer function has poles in

$$s_1 = -2 + i$$
$$s_2 = -2 - i$$

Since all poles have negative part the system is input-output stable.

Another way of checking stability of a second order system with characteristic equation $s^2 + a_1 s + a_2$ is that $a_1, a_2 > 0$.

c. Since the system is stable, the L_2 gain is given by the supremum of the transfer function gain, so we want to find the peak of the Bode amplitude plot.

```
>> s = tf('s');
>> G = 1 / (s^2 + 4*s + 5);
>> bode(G)
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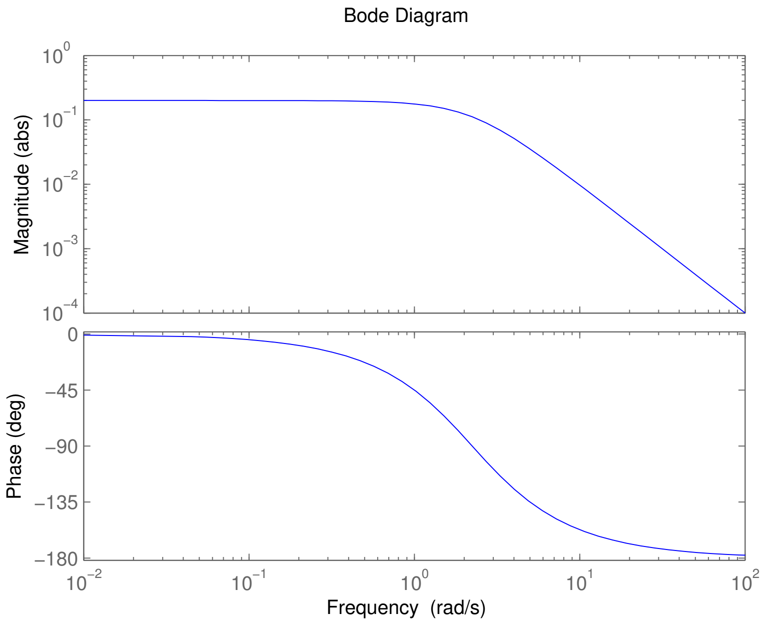

Figure 2.1 Bode diagram for Problem 2.3(c).

Alternatively, one can find the frequency that maximizes the gain by the following reasoning: Since it is a second order system, it can be written as

$$G_c(s) = \frac{K}{s^2 + 2\zeta\omega s + \omega^2}$$

In our case $\zeta = 2/\sqrt{5} \approx 0.9$. This means that the system is well damped and that it does not have a resonance peak in the gain curve. Since the gain is decreasing with frequency, the maximum gain can thus be found at $\omega = 0$.

$$|G_c(i \cdot 0)| = \frac{1}{5}$$

2.4 a.

$$S(s) = \frac{1}{1 + CP} = \frac{s^3 + 2s^2 + s}{s^3 + 2s^2 + 2.4s + 1.4} = \\ = \frac{(s+1)(s^2+s)}{(s+1)(s^2+s+1.4)} = \frac{s^2+s}{s^2+s+1.4}$$

Remark: Notice that we have a 3rd order system (with a 1st order controller and 2nd order plant), but the transfer functions $S(s)$ is only of 2nd order! Looking at the block-diagram of the system one can clearly see the pole–zero cancellation of the term $(s+1)$ for $P \cdot C$. These kind of pole-zero cancellations imply loss of either observability or loss of controllability, which will be studied later in the course.

$$T(s) = \frac{CP}{1+CP} = \frac{1.4(s+1)}{s^3+2s^2+2.4s+1.4} = \frac{1.4}{s^2+s+1.4}$$

Remark: Also for T there has been a pole-zero cancellation of $(s+1)$, but a corresponding cancellation **does not** appear in for instance $G_{d->z} = \frac{P}{1+PC}$.

The transfer function from n to z is $T(s)$ (the minus sign can be ignored since we could just as well say that the unknown noise is given by $-n$). This means that the reference and the measurement noise have the same effect on the output.

b. We know that $S(s)$ is the transfer function from load disturbance to output. Since the control system should remove the effects of load disturbances, which often are of low frequency character, it would seem reasonable if the curve representing $S(s)$ decreases as we move to the left. This corresponds to the upper curve.

We could also look at the function $S(s)$ that we just determined. We see that

$$\lim_{s\to 0} S(s) = 0$$

Comparing with the upper curve, which has a gain that goes to zero for low frequencies, we conclude that this represents the sensitivity function.

c. In order to have good tracking of the reference value, we want the gain from reference to output to be close to one. Looking at the gain curve of the *complimentary transfer function* T we see that for $\omega < 1$, we have $T \approx 1$, resulting in good tracking of the reference value.

Additional comments: At the same time, we want to be insensitive to process noise and measurement noise, i.e. we want the gain to be as small as possible for these two signals.

The transfer function from process noise to output is S, while T is the transfer function of both reference values and measurement noise to the output. S and T can not be small at the same frequencies, due to the fact that

$$S(s) + T(s) = \frac{1}{1+C(s)P(s)} + \frac{C(s)P(s)}{1+C(s)P(s)} = 1$$

Thus, we need to think about the frequency character of these signals, and compare with the shapes of the transfer functions: Process noise and reference signals are often of low frequency, so we want to have $S \approx 0$ and $T \approx 1$ at low frequencies. Measurement noise is most often of high frequency, so we want to have $T \approx 0$ at high frequencies.

d. At $\omega > 1$ T is small, resulting in good attenuation of measurement noise. (Do you see how the “speed” of control relates to the impact of measurement noise?)

2.5 **a.** The sensitivity function is given by $S = \frac{1}{1+PC}$, so S is small at frequencies where PC is large. The stationary gain of P is finite. C_2 and C_3 both have integral action and infinite stationary gain. Thus, for these controllers, S will go to zero as $\omega \to 0$. C_1, being a pure P-controller, has a finite stationary gain. S will then also have a finite stationary gain.

C_2 and C_3 are PI-controllers, but C_3 has a delay which will introduce extra phase loss. This decreases the phase margin and therefore introduces a higher sensitivity peak. Thus, we have: $C_1 \to A$, $C_2 \to C$, and $C_3 \to B$.

b. Since controller C_1 does not have integral action, we will get a stationary error in the response to a constant load disturbance, d. The response using the delayed controller C_3 will be less damped than the response using the PI-controller because of the smaller phase margin, C_2. This gives: $C_1 \to II$, $C_2 \to I$, and $C_3 \to III$.

2.6 a.

$$y = \alpha h_2, \qquad f = \beta(h_1 - h_2)$$
$$\dot{h}_1 = \frac{1}{A_1}(u_1 - f), \qquad \dot{h}_2 = \frac{1}{A_2}(u_2 + f - y)$$

$$\dot{h} = \begin{pmatrix} -\frac{1}{A_1}\beta & \frac{1}{A_1}\beta \\ \frac{1}{A_2}\beta & -\frac{1}{A_2}(\beta + \alpha) \end{pmatrix} h + \begin{pmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{pmatrix} u$$
$$y = \begin{pmatrix} 0 & \alpha \end{pmatrix} h$$

b.

$$G(s) = \frac{1}{s^2 + (2\beta + \alpha)s + \alpha\beta} \begin{pmatrix} \alpha\beta & \alpha(s + \beta) \end{pmatrix}$$

c. Since the system is stable, the L_2 gain can be computed in Matlab as:

```
>> s = tf('s');
>> G = 1/(s^2+3*s+1)*[1 s+1];
>> P = norm(G, inf)
```

The L_2 gain is $\sqrt{2}$.

d. The problem is that if v is a signal corresponding to a mass flow, then the 2-norm of that signal does not correspond to total accumulated mass flow:

$$||v||_2 = \sqrt{\int_{-\infty}^{\infty} |v(t)|^2 dt} \qquad \neq \qquad \text{total mass flow} \; = \int_{-\infty}^{\infty} v(t)dt.$$

Compare the following signals:

$$v_1(t) = \begin{cases} 1 & \text{if } 0 \le t \le 1 \\ 0 & \text{if } t > 1 \end{cases}$$
$$v_2(t) = \begin{cases} 2 & \text{if } 0 \le t \le 0.5 \\ 0 & \text{if } t > 0.5, \end{cases}$$

both corresponded to a total flow of 1 unit of mass, however their L_2-norms are different, 1 respectively $\sqrt{2}$.

In many contexts, however not quite in this one, the L_2-norm has a natural interpretation as the square root of signal energy.