Solutions to Exam in Multivariable Control 2012-10-24

1. Start by applying the Laplace transformation to both sides of the equations:

$$s^2Y_1 + 2sY_1 - Y_2 + Y_1 = U_1 + sU_2 + 2U_2$$
$$Y_2 + sY_1 + Y_1 = U_1 - 2U_2$$

or equivalently in matrix form:

$$\begin{pmatrix} (s+1)^2 & -1 \\ s+1 & 1 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} 1 & s+2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

The transfer function matrix is then given by

$$\begin{aligned} G(s) &= \begin{pmatrix} (s+1)^2 & -1 \\ s+1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & s+2 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{(s+1)(s+2)} \begin{pmatrix} 1 & 1 \\ -(s+1) & (s+1)^2 \end{pmatrix} \begin{pmatrix} 1 & s+2 \\ 1 & -2 \end{pmatrix} \\ &= \frac{1}{(s+1)(s+2)} \begin{pmatrix} 2 & s \\ s(s+1) & -3(s+1)(s+\frac{4}{3}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{s}{s+2} & -\frac{3s+4}{s+2} \end{pmatrix} \end{aligned}$$

2 a. The determinant of $G(s)$ is given by

$$\det G(s) = \frac{\alpha}{(s+1)(s+2)} + \frac{s+2}{(s+1)^2} = \frac{s^2 + (4+\alpha)s + 4 + \alpha}{(s+1)^2(s+2)}$$

Thus, the poles are located in $s = -1$ (multiplicity 2) and $s = -2$ (multiplicity 1). The transmission zeros are located at the roots of the zero polynomial

$$s^2 + (4+\alpha)s + 4 + \alpha$$

The system is non-minimum phase when $4 + a < 0$, i.e. when $a < -4$.

b. A diagonal state space realization can be derived by noting that

$$\begin{aligned} \begin{pmatrix} \frac{\alpha}{s+1} & -\frac{s+2}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{pmatrix} &= \begin{pmatrix} \frac{\alpha}{s+1} & -1 - \frac{1}{s+1} \\ \frac{1}{s+1} & \frac{1}{s+2} \end{pmatrix} \\ &= \frac{1}{s+1} \begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{s+2} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \\ &= \frac{1}{s+1} \begin{pmatrix} \alpha & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{s+2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} + D \\ &= \frac{1}{s+1} C_1 B_1 + \frac{1}{s+2} C_2 B_2 + D \end{aligned}$$

Note that the pole in $s = -1$ with multiplicity 2 requires us to have a B_1 with two columns. Also note that the factorizations $C_i B_i$ are not at all unique.

Taken together, this allows us to write the corresponding diagonal state space realization as

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} -p_1 I_{2x2} & 0 \\ 0 & -p_2 \end{pmatrix} x + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\
y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} x + D
\end{aligned}
$$

where $p_1 = 1$ and $p_2 = 2$, or explicitly

$$
\begin{aligned}
\dot{x} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u \\
y &= \begin{pmatrix} \alpha & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix} x + \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}
\end{aligned}
$$

3 a. It is a weighting function giving the unstructured uncertainty some structure. In this case it says that the uncertainty is small for low frequencies, less than a, and large for high frequencies.

b. By calling the input to the uncertainty block v_1 and the output from it v_2, the system can be rewritten according to Figure 1. To derive $G(s)$, first calculate $Y(s)$:

$$Y(s) = P(s)(V_1 - C(s)Y(s))$$

$$\Rightarrow Y(s) = \frac{P(s)}{1 + C(s)P(s)} V_1(s)$$

Now $V_2(s)$ is given by

$$V_2(s) = \frac{s}{s+a} Y(s) = \frac{P(s)\frac{s}{s+a}}{1 + C(s)P(s)} V_1(s)$$

and the transfer function $G(s)$ is hence given by

$$G(s) = \frac{P(s)\frac{s}{s+a}}{1 + C(s)P(s)}$$

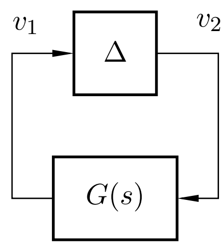

Figure 1 Equivalent block diagram for the system in problem 3.

The closed loop system is stable according to the small gain theorem if and only if $\|\Delta\| < \gamma^*$ where $\gamma^* = \frac{1}{\sup_\omega |G(i\omega)|}$

4 a. We will here refer to the state in the system P as x_1. The output can be written as $y = x_1 + v_2$ where

$$X_1(s) = \frac{6}{s+2.5}(U(s) + V_1(s)) \Rightarrow \dot{x_1} = -2.5x_1 + 6u + 6v_1$$

$$\Phi_{v_1}(\omega) = \frac{1}{\omega^2+1} \Rightarrow v_1 = \frac{1}{s+1}e_1 \Rightarrow \dot{v_1} = -v_1 + e_1 \qquad (\Phi_{e_1}(\omega) = 1)$$

In matrix form, with $x = \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}$, this becomes:

$$\begin{aligned} \dot{x} &= \begin{pmatrix} -2.5 & 6 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 6 \\ 0 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e_1 \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x + v_2 \end{aligned}$$

b. The optimal Kalman filter gain is $K = (PC^T + NR_{12})R_2^{-1}$. First, the choice of state variables (and subsequent state space representation, including C) which corresponds to the P-matrix in the hint must be found. The state representation found in the first subproblem is inserted into the Riccati equation

$$\begin{aligned} 0 &= R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T \\ \Rightarrow 0 &= R_1 + AP + PA^T - PC^TCP^T \end{aligned}$$

where $R_1 = \begin{pmatrix} 0 & 1 \end{pmatrix}^T \times 1 \times \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, R_2 = 1, R_{12} = 0$

If the first subproblem has been solved as above, this becomes

$$\begin{aligned} 0 = &\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2.5 & 6 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} + \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix}\begin{pmatrix} -2.5 & 0 \\ 6 & -1 \end{pmatrix} \\ &- \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{3}{8} \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} - \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} \end{pmatrix} = 0 \end{aligned}$$

and we have therefore showed that the given P-matrix relates to the state representation $X = \begin{pmatrix} x_1 \\ v_1 \end{pmatrix}$, for which we have known A, B and C matrices. If the state representation is inverted, i.e. $X = \begin{pmatrix} v_1 \\ x_1 \end{pmatrix}$, the Riccati equation will not hold and it can be realized that the only other state representation is the aforementioned one, for which we can show as above that the Riccati equation holds.

Using the C matrix given by this state representation, we can then calculate K using the (simplified) expression

$$K = PC^T = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

5.

1. True. The Bode diagram clearly shows that there is an integrator in the process, meaning that for low frequencies the sensitivity function $S(s) = 1/(1 + P(s)C(s))$ will be close to zero. In particular, $S(0) = 0$.
2. False. Because of the integrator in the process, the transfer function from input load disturbance to output at stationarity $P(0)S(0)$ will not be zero.
3. True. According to the Bode diagram of the process, we need to add phase at the desired cut-off frequency to get the specified phase margin. A PI-controller can never give a net increase in phase, so this will not be possible.
4. False. Assuming that the closed loop system is stable, the final value theorem says that for the error

$$\begin{aligned}\lim_{t\to\infty} e(t) &= \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{1}{1+P(s)C(s)}\frac{1}{s^2} \\ &= \lim_{s\to 0} \frac{1}{s + sP(s)C(s)} = C \neq 0\end{aligned}$$

   where C is a nonzero constant, as the integrator in $P(s)$ will be canceled out by the factor $1/s$. Thus, there will be a stationary error.

6. This problem can either be solved by studying the A, B and C matrices directly or through calculation of the observability and controllability matrices.

Using the first approach:
The dynamics for all states in the system differ from each other (there are no multiple poles), therefore a state is controllable if the control signals can influence its value (directly or indirectly) and observable if its value can influence the measurement signals (directly or indirectly). This means that states 1, 2, 3 and 4 are controllable and states 1, 2, 4 and 5 are observable. The system as a whole is therefore neither controllable nor observable and the controllable subspace consists of all states except the 5th and the observable subspace consists of all states except the 3rd.

Using the second approach:

$$S = \begin{pmatrix} B & AB & \dots & A^{n-1}B \end{pmatrix} \qquad (n = 5)$$

$$\Rightarrow$$

$$S = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -2 & 0 & 8 & 0 & -26 & 0 & 80 \\ 1 & 0 & -4 & 0 & 16 & 0 & -64 & 0 & 256 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The controllability matrix has rank lower than 5 so the system is not controllable. We can see that the 5th state is not part of the controllable subspace, as the 5th row in the controllability matrix only contains zeros.

$$O = \begin{pmatrix} C \\ CA \\ AB \\ \vdots \\ CA^{n-1} \end{pmatrix} \qquad (n = 5)$$

$\Rightarrow$

$$O = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & -5 \\ -2 & -3 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 25 \\ 8 & 9 & 0 & 0 & 25 \\ 0 & 0 & 0 & 0 & -125 \\ -26 & -27 & 0 & 0 & -125 \\ 0 & 0 & 0 & 0 & 625 \\ 80 & 81 & 0 & 0 & 625 \end{pmatrix}$$

The observability matrix has rank lower than 5 so the system is not observable. We can see that the 3rd state is not part of the observable subspace, as the 3rd column in the observability matrix only contains zeros.

7 a. The controllability gramian S and the observability gramian O are given by the solution to the Lyapunov equations

$$AS + SA^T + BB^T = 0$$
$$A^T O + OA + C^T C = 0$$

Since $A = A^T$ and $B = C^T$, this reduces to only solving one of the Lyapunov equations. If the realization is balanced, this amounts to finding a solution in the form

$$S = O = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$$

The terms of the Lyapunov equation then gives the following set of equations

$$-4\sigma_1 + \frac{1}{4} = 0$$
$$\sigma_1 + \sigma_2 + \frac{1}{2}(-1 - \frac{\sqrt{2}}{2}) = 0$$
$$-4\sigma_2 + \frac{1}{4} + (-1 - \frac{\sqrt{2}}{2})2 = 0$$

with solution $\sigma_1 = 1/16 = 0.0625$ and $\sigma_2 = \frac{7+4\sqrt{2}}{16} \approx 0.7911$. Hence, the realization is balanced.

b. The smallest Hankel singular value is σ_1. This corresponds to eliminating ξ_1:

$$0 = -2\xi_1 + \xi_2 + \frac{1}{2}y_2 \implies$$
$$\xi_1 = \frac{1}{2}\xi_2 + \frac{1}{4}y_2$$

Inserting this into the rest of the system equations gives

$$\begin{aligned}
\dot{\xi}_2 &= \xi_1 - 2\xi_2 + \frac{1}{2}y_1 + (-1 - \frac{\sqrt{2}}{2})y_2 \\
&= \frac{1}{2}\xi_2 + +\frac{1}{4}y_2 - 2\xi_2 + \frac{1}{2}y_1 + (-1 - \frac{\sqrt{2}}{2})y_2 \\
&= -\frac{3}{2}\xi_2 + \frac{1}{2}y_1 - \frac{3+2\sqrt{2}}{4}y_2 \\
u_1 &= \frac{1}{2}\xi_2 \\
u_2 &= \frac{1}{2}\xi_1 + (-1 - \frac{\sqrt{2}}{2})\xi_2 = \frac{1}{4}\xi_2 + \frac{1}{8}y_2 + (-1 - \frac{\sqrt{2}}{2})\xi_2 \\
&= -\frac{3+2\sqrt{2}}{4}\xi_2 + \frac{1}{8}y_2
\end{aligned}$$

or in matrix form

$$\begin{aligned}
\dot{\xi}_2 &= -\frac{3}{2}\xi_2 + \begin{pmatrix} \frac{1}{2} & -\frac{3+2\sqrt{2}}{4} \end{pmatrix} y = A\xi_2 + By \\
u &= \begin{pmatrix} \frac{1}{2} \\ -\frac{3+2\sqrt{2}}{4} \end{pmatrix} \xi_2 + \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{8} \end{pmatrix} y = C\xi_2 + Dy
\end{aligned}$$

c. Using the Laplace transform and some calculations, the transfer matrix is given by

$$G(s) = C(sI - A)^{-1}B + D = \frac{1}{s+1.5}\begin{pmatrix} \frac{1}{4} & -\frac{3+2\sqrt{2}}{8} \\ -\frac{3+2\sqrt{2}}{8} & \frac{s+10+6\sqrt{2}}{8} \end{pmatrix}$$

8. A: The unstable pole ($p = 2$) and the unstable zero ($z = 3$) means that $\|S\|_\infty \geq \left|\frac{z+p}{z-p}\right| = 5$), and as the given plot of S is less than or equal to 1 it is clear that the control design is not possible.

B: The unstable pole means that the closed loop system needs a bandwidth that is at least as fast as the pole. The system in the plot has a bandwidth of approximately 50 rad/s, and the design may hence be possible.

C: The unstable zero makes it impossible to achieve a closed loop bandwidth larger than 1 rad/s, which means that the design in the plot that has a bandwidth of above 20 rad/s is not possible.

D: By using the reverse triangle inequality: $|S - T| \geq |(|S| - |T|)|$, the following inequality can be derived:

$$1 = |1| = |S + T| = |S - (-T)| \geq |(|S| - |T|)|$$

$$\Rightarrow |(|S| - |T|)| \leq 1$$

At the frequency $\omega \approx 3$ rad/s it can be seen in the plot that $|S| = 4$ and $|T| = 0.7$, but inserting this into the above relation gives $|(|S| - |T|)| = |4 - 0.7| = 3.3$, and this is clearly not less than 1, and this design is therefore also not possible.