Lecture 15: Course Summary

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Examples

Flexible servo resonant system

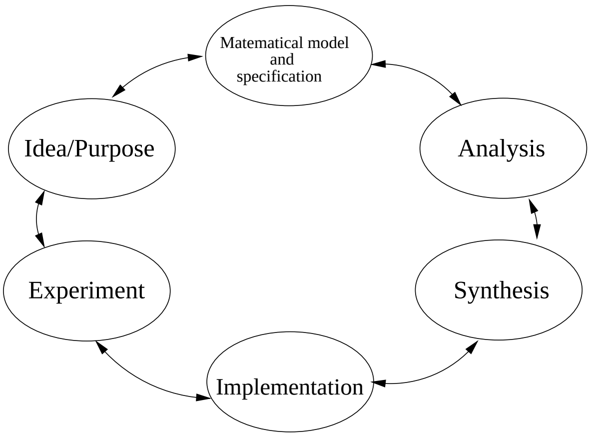
Quadruple tank system multivariable (MIMO), NMP-zero

Rotating crane multivariable, observer needed

DVD control resonant system, wide frequency range, (midranging)

Bicycle steering unstable pole/zero-pair

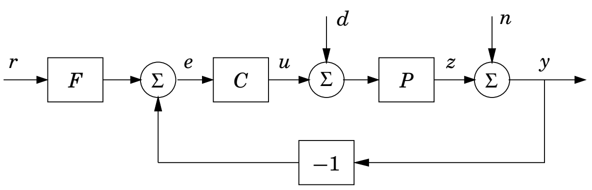
Distillation column MIMO, input-output pairing



Course Summary

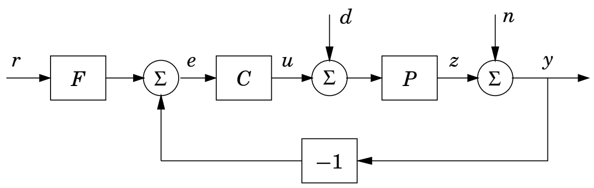
- **Specifications, models and loop-shaping**
- Limitations on achievable performance
- Controller optimization: Analytic approach
- Controller optimization: Numerical approach

2DOF control



- Reduce the effects of load disturbances
- Limit the effects of measurement noise
- Reduce sensitivity to process variations
- Make output follow command signals

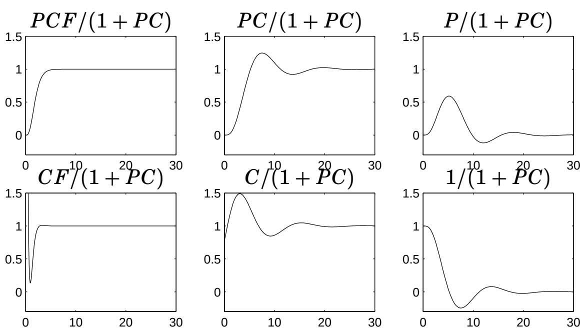
2DOF control



$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

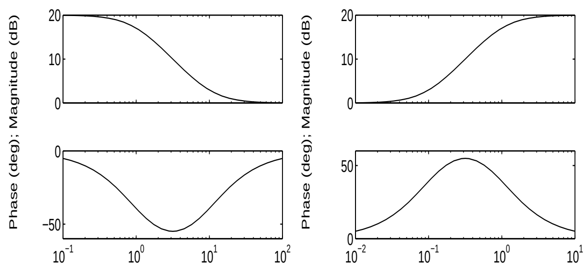
$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

Important Step Responses



Lag and lead filters for loop-shaping of $P(s)C(s)$

$$C(s) = \frac{s+10}{s+1} \qquad C(s) = \frac{10(s+1)}{(s+10)}$$



MIMO-systems

If C, P and F are general MIMO-systems, so called *transfer function matrices*, the **order of multiplication matters** and

$$PC \neq CP$$

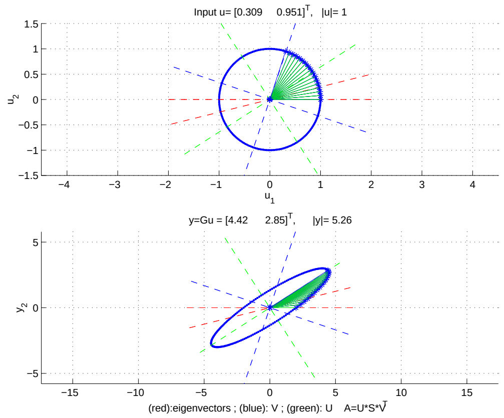
and thus we need to multiply with the inverse from the correct side as in general

$$(I+L)^{-1}M \neq M(I+L)^{-1}$$

Note, however that

$$(I+PC)^{-1}PC = P(I+CP)^{-1}C = PC(I+PC)^{-1}$$

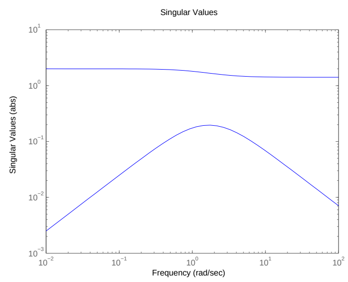
Different gains in different directions: $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$



Plot Singular Values of $G(i\omega)$ Versus Frequency

```
» s=tf('s')
» G=[1/(s+1) 1 ; 2/(s+2) 1]
» sigma(G) % plot singular values
```

```
% Alt. for a certain frequency:
» w = 1;
» A = [1/(i*w+1) 1; 2/(i*w+2) 1]
» [U,S,V] = svd(A)
```



Realization of Multi-variable system

Example: To find state space realization for the system

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{2}{(s+1)(s+3)} \\ \frac{6}{(s+2)(s+4)} & \frac{1}{s+2} \end{bmatrix}$$

write the transfer matrix as

$$\begin{bmatrix} \frac{1}{s+1} & \frac{1}{s+1} - \frac{1}{s+3} \\ \frac{3}{s+2} - \frac{3}{s+4} & \frac{1}{s+2} \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix}}{s+1} + \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \end{bmatrix}}{s+2} - \frac{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}}{s+3} - \frac{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \end{bmatrix}}{s+4}$$

This gives the realization

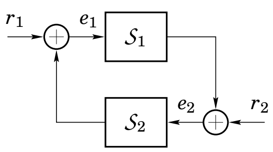
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \\ 0 & -1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} x(t)$$

The Small Gain Theorem

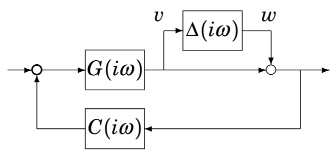
Consider a system $\mathcal{S}$ with input u and output $\mathcal{S}(u)$ having a (Hurwitz) stable transfer function $G(s)$. Then, the system gain

$$\|\mathcal{S}\| := \sup_u \frac{\|\mathcal{S}(u)\|}{\|u\|} \quad \text{is equal to} \quad \|G\|_\infty := \sup_\omega |G(i\omega)|$$



Assume that $\mathcal{S}_1$ and $\mathcal{S}_2$ are input-output stable. If $\|\mathcal{S}_1\| \cdot \|\mathcal{S}_2\| < 1$, then the gain from (r_1, r_2) to (e_1, e_2) in the closed loop system is finite.

Application to robustness analysis



The transfer function from w to v is

$$\frac{G(i\omega)C(i\omega)}{1+G(i\omega)C(i\omega)}$$

Hence the small gain theorem guarantees closed loop stability for all perturbations Δ with

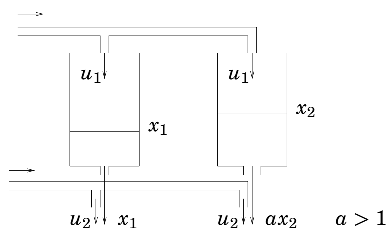
$$\|\Delta\| < \left(\sup_\omega \left| \frac{G(i\omega)C(i\omega)}{1+G(i\omega)C(i\omega)} \right| \right)^{-1}$$

Course Summary

- Specifications, models and loop-shaping
- **Limitations on achievable performance**
- Controller optimization: Analytic approach
- Controller optimization: Numerical approach

Example: Two water tanks

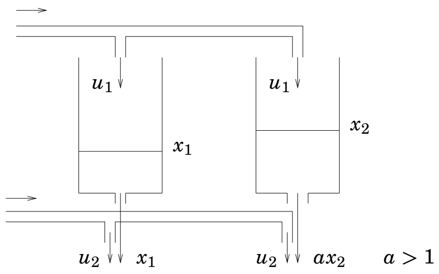
Example from Lecture 6:



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$
$$y_1 = x_1 + u_2 \qquad y_2 = ax_2 + u_2$$

Can you reach $y_1 = 1, y_2 = 2$? Can you stay there?

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

The controllability Gramian $S = \int_0^\infty \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix} \begin{bmatrix} e^{-t} \\ e^{-at} \end{bmatrix}^T dt = \begin{bmatrix} \frac{1}{2} & \frac{1}{a+1} \\ \frac{1}{a+1} & \frac{1}{2a} \end{bmatrix}$

is close to singular for $a \approx 1$, so it is harder to reach a desired state.

Computing the controllability Gramian

The controllability Gramian $S = \int_0^\infty e^{At} B B^T e^{A^T t} dt$ can be computed by solving the linear system of equations

$$AS + SA^T + BB^T = 0$$

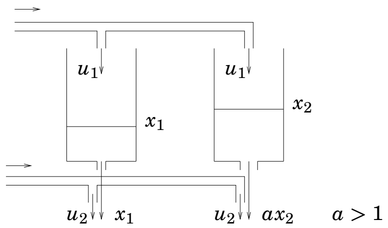
$S = S^T > 0$, i.e., S is a symmetric positive definite matrix

Assign

$$S = \begin{bmatrix} s_{11} & s_{12} \\ s_{12} & s_{22} \end{bmatrix}$$

Multiply together and solve for s_{11}, s_{12}, s_{22} in the same way as you also do for the spectral factorization and the Riccati equations...

Example: Two water tanks



$$\dot{x}_1 = -x_1 + u_1 \qquad \dot{x}_2 = -ax_2 + u_1$$

$G(s) = \begin{bmatrix} \frac{1}{s+1} & 1 \\ \frac{2}{s+2} & 1 \end{bmatrix}$. Find zero from $\det G(s) = \dfrac{-s}{(s+1)(s+2)}$

There is a zero at $s = 0$! Outputs must be equal at stationarity.

Sensitivity bounds from RHP zeros and poles

Rules of thumb:

"The closed-loop bandwidth must be less than z."
"The closed-loop bandwidth must be greater than p."
"Time delays T must be less than $1/p$."

Hard bounds:

The sensitivity must be one at an unstable zero:

$$P(z) = 0 \qquad \Rightarrow \qquad S(z) := \frac{1}{1 + P(z)C(z)} = 1$$

The complimentary sensitivity must be one at an unstable pole:

$$P(p) = \infty \qquad \Rightarrow \qquad T(p) := \frac{P(p)C(p)}{1 + P(p)C(p)} = 1$$

Maximum Modulus Theorem

Assume that $G(s)$ is rational, proper and stable. Then

$$\max_{\text{Re } s \geq 0} |G(s)| = \max_{\omega \in \mathbf{R}} |G(i\omega)|$$

Corollary:

Suppose that the plant $P(s)$ has unstable zeros z_i and unstable poles p_j. Then the specifications

$$\sup_\omega |W_a(i\omega)S(i\omega)| < 1 \qquad \sup_\omega \left|W^b(i\omega)T(i\omega)\right| < 1$$

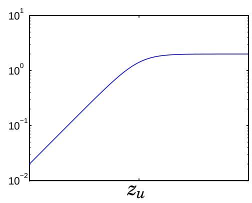
are impossible to meet with a stabilizing controller unless $\|W_a(z_i)\| < 1$ for every unstable zero z_i and $\|W^b(p_j)\| < 1$ for every unstable pole p_j.

Hard limitations from unstable zeros

If the plant has an unstable zero z_u, then the specification

$$\left|\frac{1}{1 + P(i\omega)C(i\omega)}\right| < \frac{2}{\sqrt{1 + z_u^2/\omega^2}} \qquad \text{for all } \omega$$

is impossible to satisfy.



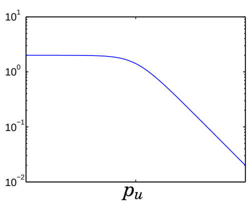
Examples: Rear-wheel steering and quadruple tank process

Hard limitations from unstable poles

If the plant has an unstable pole p_u, then the specification

$$\left|\frac{P(i\omega)C(i\omega)}{1 + P(i\omega)C(i\omega)}\right| < \frac{2}{\sqrt{\omega^2/p_u^2 + 1}} \qquad \text{for all } \omega$$

is impossible to satisfy.



Example: Inverted pendulum

Nonmin-phase zero and unstable pole

Let $P = \widehat{P}(s-z)(s-p)^{-1}$, with $\widehat{P}$ proper and $\widehat{P}(p) \neq 0$.

Then, for stable closed loop the sensitivity function satisfies

$$\sup_\omega |S(i\omega)| \geq \left|\frac{z+p}{z-p}\right|$$

so if $p \approx z$, then the sensitivity function must have a high peak *for every controller* C.

Example: Bicycle with rear wheel steering

$$\frac{\theta(s)}{\delta(s)} = \frac{am\ell V_0}{bJ} \cdot \frac{(-s + V_0/a)}{(s^2 - mg\ell/J)}$$

Relative Gain Array (RGA)

For a square matrix $A \in \mathbf{C}^{n \times n}$, define

$$\text{RGA}(A) := A .* (A^{-1})^T$$

where ".*" denotes element-by-element multiplication.
(For a non-square matrix, use pseudo inverse $A^{\dagger}$)

- The sum of all elements in a column or row is one.
- Permutations of rows or columns in A give the same permutations in RGA(A)
- RGA(A) = RGA$(D_1 A D_2)$ if D_1 and D_2 are diagonal, i.e. RGA(A) is independent of scaling
- If A is triangular, then RGA(A) is the unit matrix I.

RGA for a Distillation Column

- Find a permutation of inputs and outputs that makes RGA$(P(0))$ as close as possible to the identity matrix.
- Avoid pairings that give negative diagonal elements of RGA$(P(0))$

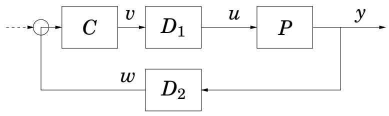
$$\text{RGA}(P(0)) = \begin{bmatrix} 0.2827 & -0.6111 & 1.3285 \\ 0.0134 & 1.5827 & -0.5962 \end{bmatrix}$$

To choose control signal for y_1, we apply the heuristics to the top row and choose u_3. Based on the bottom row, we choose u_2 to control y_2. Decentralized control!

Decoupling

Simple idea: Find a compensator so that the system appears to be without coupling ("block-diagonal transfer function matrix").

- Input decoupling $Q = PD_1$
- Output decoupling $Q = D_2 P$
- "both" $Q = D_2 P D_1$



Find D_1 and D_2 so that the controller sees a "diagonal plant":

$$D_2 P D_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a "decentralized" controller C with same block-diagonal structure.

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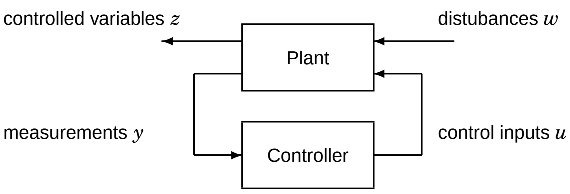
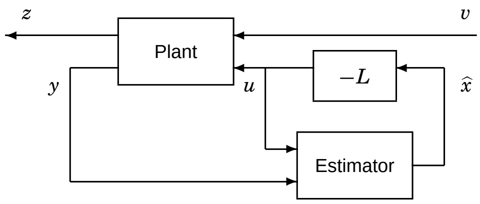
A General Optimization Setup



The objective is to find a controller that optimizes the transfer matrix $G_{zw}(s)$ from disturbances w to controlled outputs z.

Lecture 9-11: Problems with analytic solutions
Lectures 12-14: Problems with numeric solutions

Output feedback using state estimates



Plant: $\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases}$

Controller: $\begin{cases} \frac{d}{dt}\hat{x}(t) = A\hat{x}(t) + Bu(t) + K[y(t) - C\hat{x}(t)] \\ u(t) = -L\hat{x}(t) \end{cases}$

Linear Quadratic Optimal Control (LQG)

Given the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(k) \\ y(t) = Cx(t) + v_2(t) \\ z(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \end{cases} \qquad Q = \begin{bmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{bmatrix} \qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\hat{x}$ with $\frac{d}{dt}\hat{x} = A\hat{x} + Bu + K[y - C\hat{x}]$. The frequency integral

$$\text{trace} \; \frac{1}{2\pi}\int_{-\infty}^{\infty} QG_{zv}(i\omega)RG_{zv}(i\omega)^* d\omega$$

is minimized when K and L satisfy

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \qquad L = Q_2^{-1}(SB + Q_{12})^T$$

$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T \qquad K = (PC^T + R_{12})R_2^{-1}$$

The minimal value of the integral is

$$\text{tr}(SR_1) + \text{tr}[PL^T(B^T SB + Q_2)L]$$

Stochastic Interpretation of LQG Control

Given white noise (v_1, v_2) with intensity R and the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + v_1(t) \\ y(t) = Cx(t) + v_2(t) \end{cases} \qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\widehat{x}$ with $\frac{d}{dt}\widehat{x} = A\widehat{x} + Bu + K[y - C\widehat{x}]$. The stationary variance

$$\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$$

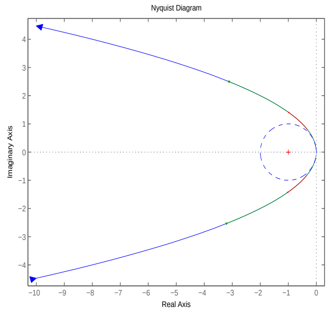
is minimized when K and L satisfy

$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \qquad L = Q_2^{-1}(SB + Q_{12})^T$$
$$0 = R_1 + AP + PA^T - (PC^T + R_{12})R_2^{-1}(PC^T + R_{12})^T \qquad K = (PC^T + R_{12})R_2^{-1}$$

The minimal variance is

$$\text{tr}(SR_1) + \text{tr}[PL^T(B^T SB + Q_2)L]$$

Stability robustness of optimal state feedback



Notice that the distance from $L(i\omega I - A)^{-1}B$ to -1 is never smaller than 1. This is always true (!) for linear quadratic optimal state feedback when $Q_1 > 0$, $Q_{12} = 0$ and $Q_2 = \rho > 0$ is scalar. Hence the phase margin is at least $60°$.

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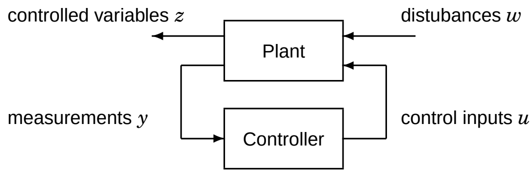
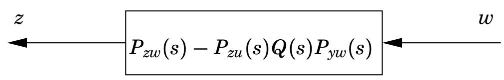
The Q-parametrization (Youla)

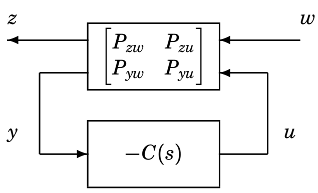


Idea for lecture 12-14:
The choice of controller generally corresponds to finding $Q(s)$, to get desirable properties of the map from w to z:



Once $Q(s)$ is determined, a corresponding controller is derived.

The Youla Parametrization



The closed loop transfer matrix from w to z is

$$G_{zw}(s) = P_{zw}(s) - P_{zu}(s)Q(s)P_{yw}(s)$$

where

$$Q(s) = C(s)\left[I + P_{yu}(s)C(s)\right]^{-1}$$
$$C(s) = Q(s) + Q(s)P_{yu}(s)C(s)$$
$$C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1}Q(s)$$

Synthesis by convex optimization

A general control synthesis problem can be stated as a convex optimization problem in the variables $Q_0, \ldots, Q_m$. The problem has a quadratic objective, with linear and quadratic constraints:

Minimize $\displaystyle\int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega)\overbrace{\sum_k Q_k\phi_k(i\omega)}^{Q(i\omega)}\, P_{yw}(i\omega)|^2 d\omega$ } quadratic objective

subject to
step response $w_i \to z_j$ is smaller than f_{ijk} at time t_k
step response $w_i \to z_j$ is bigger than g_{ijk} at time t_k } linear constraints
Bode magnitude $w_i \to z_j$ is smaller than h_{ijk} at ω_k } quadratic constraints

Once the variables $Q_0, \ldots, Q_m$ have been optimized, the controller is obtained as $C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1}Q(s)$

Model reduction by balanced truncation

Consider a balanced realization

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + Du$$

with the lower part of the gramian being $\Sigma_2 = \begin{bmatrix} \sigma_{r+1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$.

Replacing the second state equation by $\dot{\xi}_2 = 0$ gives the relation $0 = A_{21}\xi_1 + A_{22}\xi_2 + B_2 u$. The reduced system

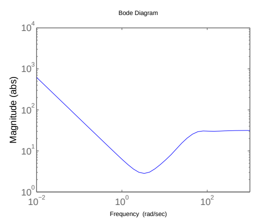
$$\begin{cases} \dot{\xi}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\xi_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_r = (C_1 - C_2A_{22}^{-1}A_{21})\xi_1 + (D - C_2A_{22}^{-1}B_2)u \end{cases}$$

satisfies the error bound

$$\frac{\|y - y_r\|_2}{\|u\|_2} \leq 2\sigma_{r+1} + \cdots + 2\sigma_n$$

DC-servo example

Recall the Bode plot of the optimized controller $C_{\text{opt}}(s)$ from Lec.14:



The Hankel singular values of $C_{\text{stab}}(s) = C_{\text{opt}}(s) + \frac{6.17}{s}$ are

```
Sigma = [16.0768   2.2306   0.7023   0.1994   0.0896]
```

Only one state needs to be kept in $C_{\text{stab}}(s)$.

What remains of $C_{\text{opt}}(s) = C_{\text{stab}}(s) - \frac{6.17}{s}$ is a PID controller.