FRTN10 Multivariable Control, Lecture 11

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Course Outline

L1-L5 Specifications, models and loop-shaping by hand

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

9. Linear quadratic optimal control
10. Optimal output feedback (LQG)
11. **More on LQG**

L12-L14 Controller optimization: Numerical approach

Recall the main result of LQG

Given white noise (v_1, v_2) with intensity R and the linear plant

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + Nv_1(k) \\ y(t) = Cx(t) + v_2(t) \end{cases} \qquad R = \begin{bmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{bmatrix}$$

consider controllers of the form $u = -L\widehat{x}$ with $\frac{d}{dt}\widehat{x} = A\widehat{x} + Bu + K[y - C\widehat{x}]$. The stationary variance

$$\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$$

is minimized when

$$K = (PC^T + NR_{12})R_2^{-1} \qquad L = Q_2^{-1}(SB + Q_{12})^T$$

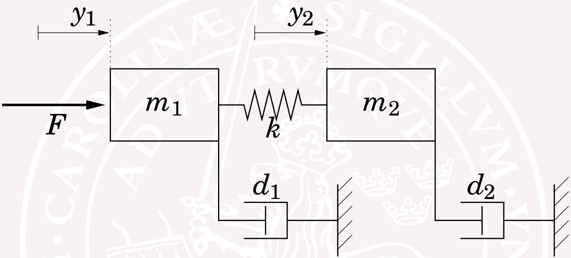
$$0 = Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T$$

$$0 = NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T$$

The minimal variance is

$$\text{tr}(SNR_1N^T) + \text{tr}[PL^T(B^T SB + Q_2)L]$$

LQG Example 1 — Flexible servo

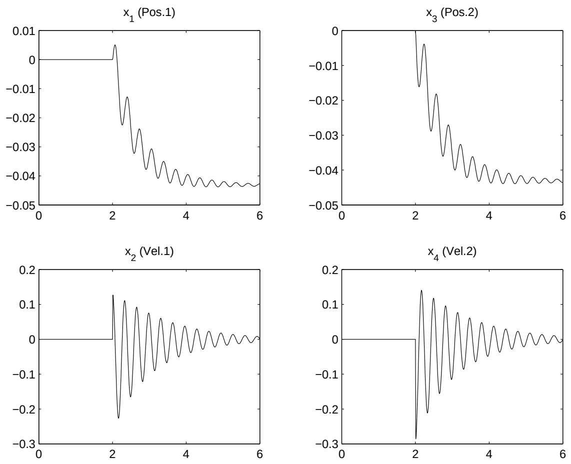


$$m_1 \frac{d^2 y_1}{dt^2} = -d_1 \frac{dy_1}{dt} - k(y_1 - y_2) + F(t)$$

$$m_2 \frac{d^2 y_2}{dt^2} = -d_2 \frac{dy_2}{dt} + k(y_1 - y_2)$$

Introduce state variables $x_1 = y_1, \; x_2 = \dot{y}_1, \; x_3 = y_2, \; x_4 = \dot{y}_2$

Open loop response



Choice of minimization criterion

How choose Q_1, Q_2, Q_{12} in the cost function

$$x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u$$

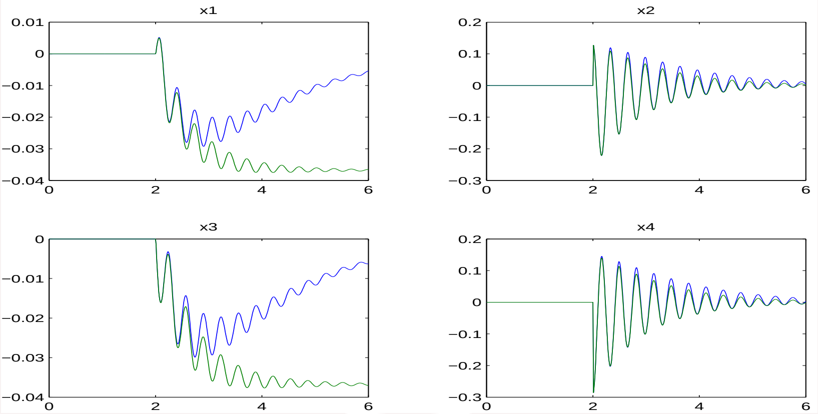
Rules of thumb:

- Put $Q_{12} = 0$ and make Q_1, Q_2 diagonal
- Make the diagonal elements equal to the inverse value of the square of the allowed deviation:

$$x(t)^T Q_1 x(t) + u(t)^T Q_2 u(t)$$
$$= \left(\frac{x_1(t)}{x_1^{\max}}\right)^2 + \cdots + \left(\frac{x_n(t)}{x_n^{\max}}\right)^2 + \left(\frac{u_1(t)}{u_1^{\max}}\right)^2 + \cdots + \left(\frac{u_m(t)}{u_m^{\max}}\right)^2$$

Penalize velocity error or position error?

Minimize $\mathbf{E}[x_2(k)^2 + x_4(k)^2 + u(k)^2]$ or $\mathbf{E}[x_1(k)^2 + x_3(k)^2 + u(k)^2]$?



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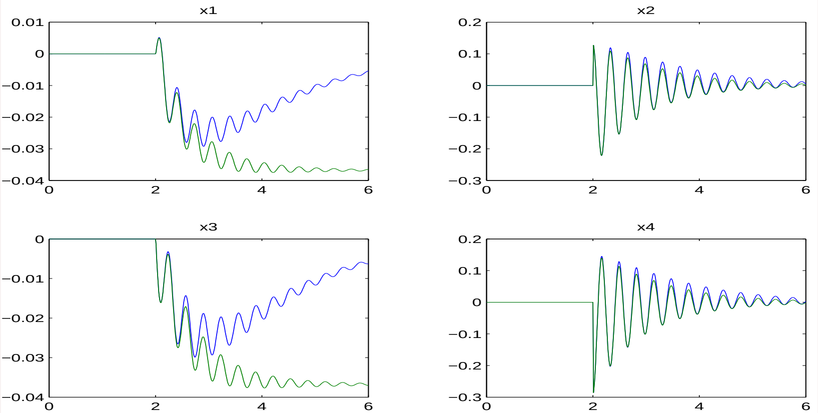
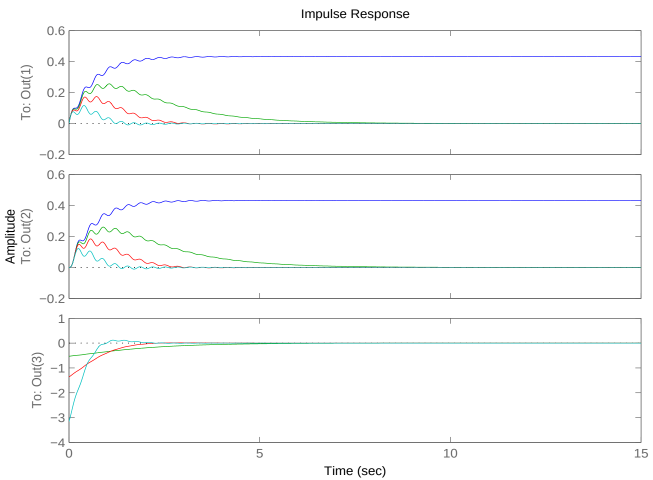


When only velocity is penalized, a static position error remains

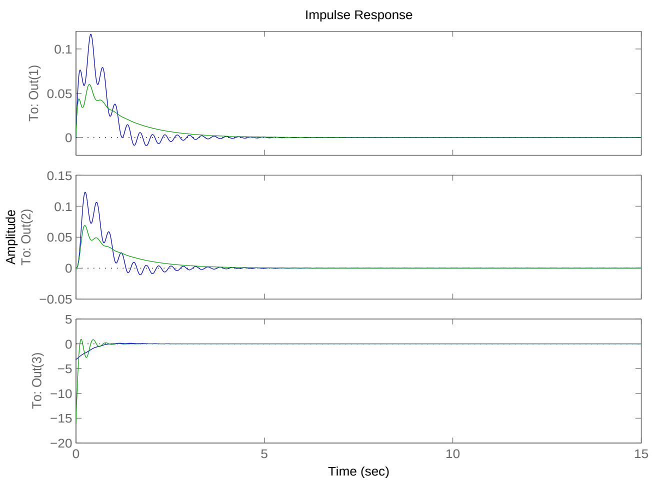
Position error control

Response of $x_1(k), x_3(k), u(k) = -Lx(k)$ to impulse disturbance.
$Q_1 = \text{diag}\{q, 0, q, 0\}$ ($q = 0, 1, 10, 100$), $Q_{12} = 0$, $Q_2 = 1$.
Large $q \Rightarrow$ fast response but large control signal.

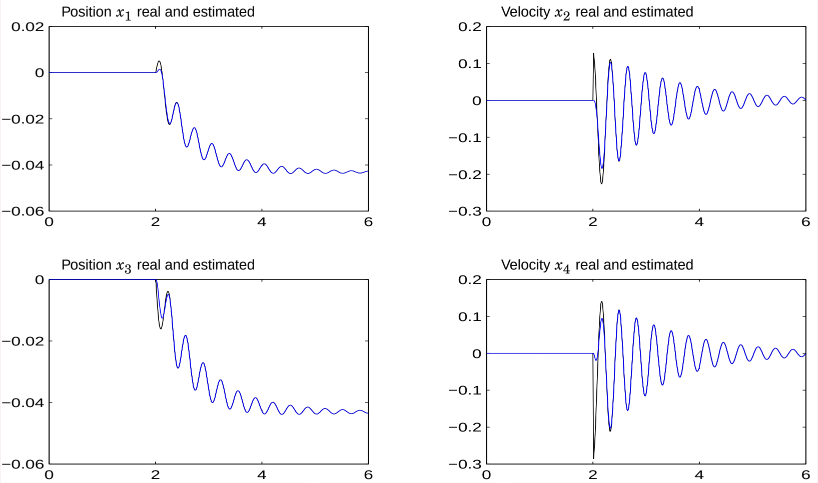


Position+velocity error control

To reduce oscillations, penalize also velocity error. Comparision between $Q_1 = \text{diag}\{100, 0, 100, 0\}$ and $Q_1 = \text{diag}\{100, 100, 100, 100\}$.



Real and estimated states



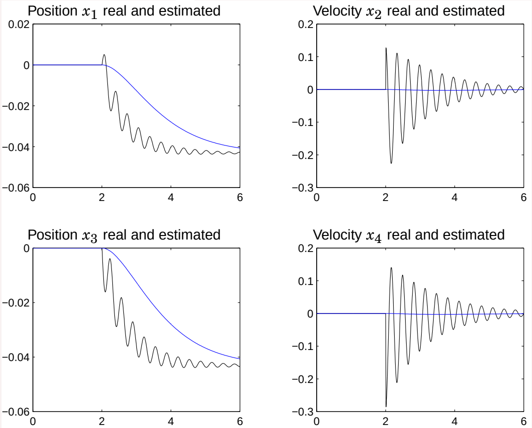
A Kalman filter estimates the states using measured positions.

Miniproblem

What happens if

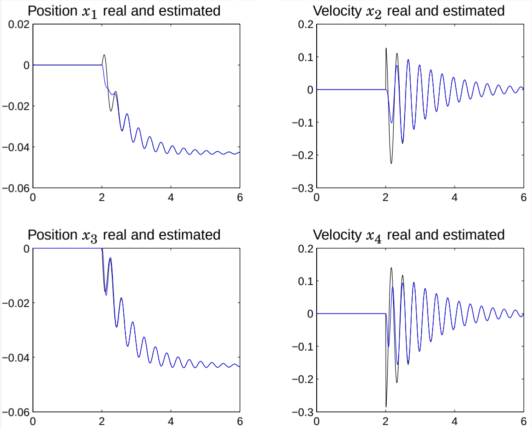
- we reduce R_1 by 10000?
- we increase the upper left corner of R_2 by 10000?
- we increase the lower right corner of R_2 by 10000?

Reduced R_1



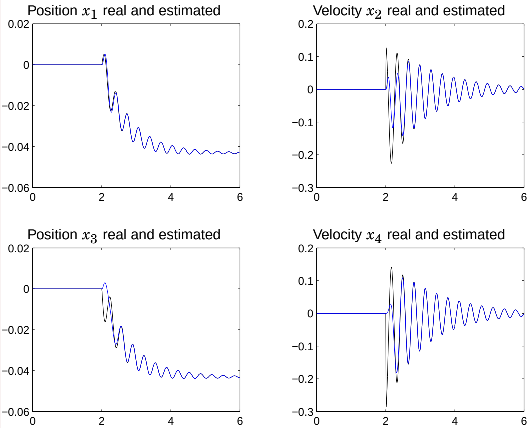
When the expected process disturbances are small, the observer will be slower.

Increased the upper left corner of R_2



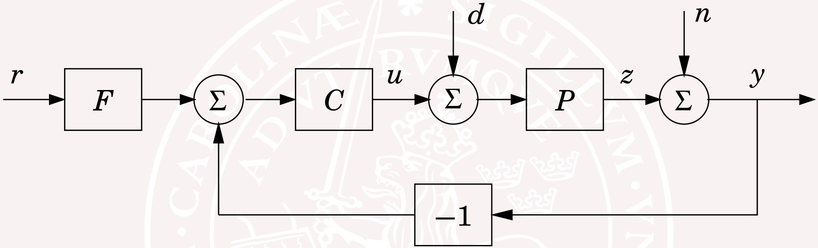
The measurement y_1 is not trusted, so the estimate of x_1 slows down.

Increased lower right corner of R_2



The measurement y_2 is not trusted, so the estimate of x_3 slows down.

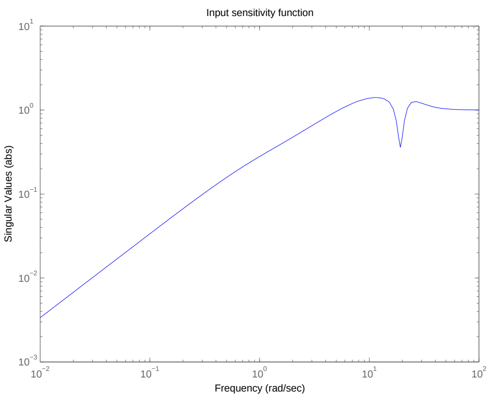
Recall the simple control loop



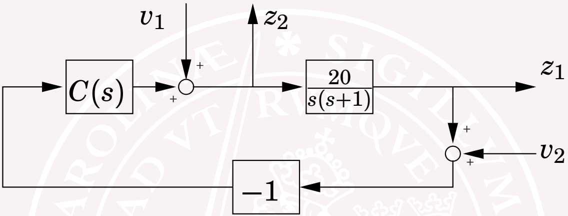
- Reduce the effects of load disturbances
- Limit the effects of measurement noise
- Reduce sensitivity to process variations
- Make output follow command signals

Don’t forget “The Gang of Four”!

Check all relevant transfer functions for robustness and signal sizes. The input sensitivity $|(I + CP)^{-1}(i\omega)|$ is plotted below. No large peaks, maximum=1.4.



LQG Example 2 — DC-servo



With $P(s) = \frac{20}{s(s+1)}$, the transfer matrix from (v_1, v_2) to (z_1, z_2) is

$$G_{zv}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \end{bmatrix}$$

As a first (preliminary) design, we choose $C(s)$ to minimize

$$\text{trace} \int_{-\infty}^{\infty} G_{zv}(i\omega) G_{zv}(i\omega)^* d\omega$$

This minimizes $\mathbf{E}(|z_1|^2 + |z_2|^2)$ when (v_1, v_2) is white noise.

LQG Design

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}}^{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \overbrace{\begin{bmatrix}20\\ 0\end{bmatrix}}^{B}u + \overbrace{\begin{bmatrix}20\\ 0\end{bmatrix}}^{N}v_1$$

$$y = x_2 + v_2 \qquad z_1 = x_2 \qquad z_2 = u + v_1$$

Minimization of $\mathbf{E}(|z_1|^2 + |z_2|^2)$ is the LQG problem defined by

$$Q_1 = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \qquad Q_2 = 1 \qquad R = \begin{bmatrix}R_1 & 0\\ 0 & R_2\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Solving the Riccati equations gives the optimal controller

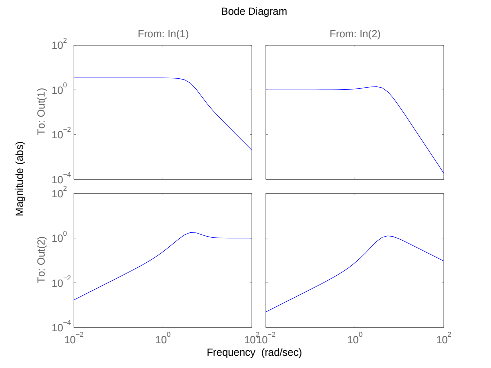
$$\frac{d}{dt}\widehat{x} = (A - BL)\widehat{x} + K[y - C\widehat{x}] \qquad u = -L\widehat{x}$$

where

$$L = \begin{bmatrix}0.2702 & 0.7298\end{bmatrix} \qquad K = \begin{bmatrix}20.0000\\ 5.4031\end{bmatrix}$$

Bode magnitude plots

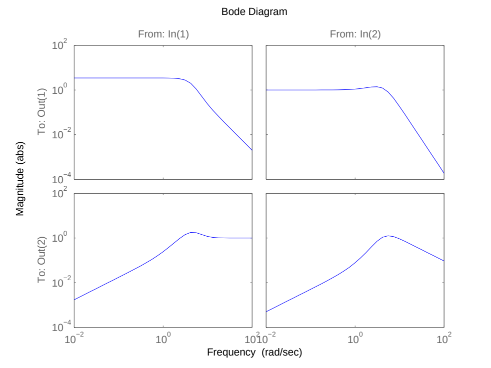
$$G_{zv}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \end{bmatrix}$$



Nonzero static gain in $\frac{P}{1+PC}$ indicates poor disturbance rejection

Bode magnitude plots

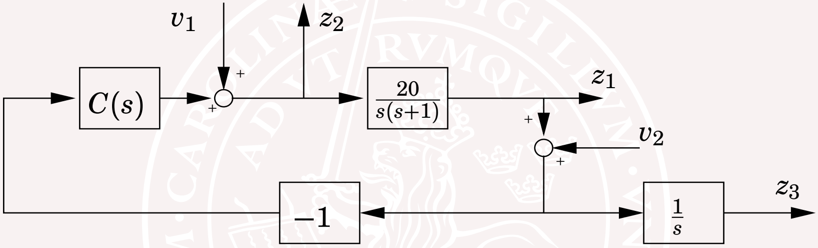
$$G_{zv}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \end{bmatrix}$$



Nonzero static gain in $\frac{P}{1+PC}$ indicates poor disturbance rejection

Integral action

To remove stationary errors in the output we penalize also z_3:



The transfer matrix from (v_1, v_2) to (z_1, z_2, z_3) is

$$G_{zv}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \\ \frac{P}{s(1+PC)} & \frac{-PC}{s(1+PC)} \end{bmatrix}$$

Extended DC-motor model

With the model

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \overbrace{\begin{bmatrix}0 & 0 & 0\\ 1 & -1 & 0\\ 0 & 1 & 0\end{bmatrix}}^{A_e}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \overbrace{\begin{bmatrix}20\\ 0\\ 0\end{bmatrix}}^{B_e} u + \overbrace{\begin{bmatrix}20 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}}^{N_e}\overbrace{\begin{bmatrix}v_1\\ v_2\end{bmatrix}}^{v_{1e}}$$

$$y = x_2 + v_2$$

minimization of $|x_2|^2 + |x_3|^2 + |u|^2$ gives the optimal controller

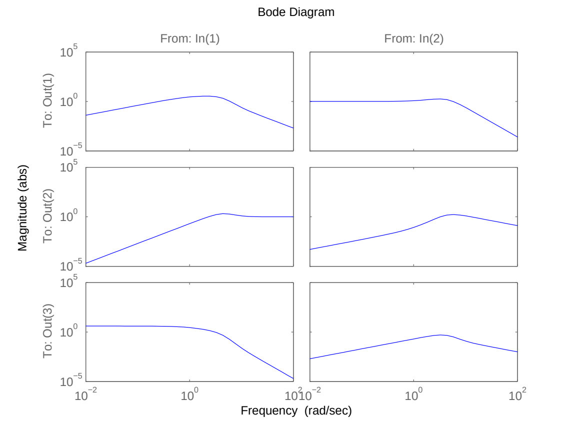
$$\frac{d}{dt}\widehat{x}_e = (A_e - B_e L_e)\widehat{x}_e + K_e[y - C_e\widehat{x}_e] \qquad u = -L\widehat{x}$$

where

$$C_e = \begin{bmatrix}0.0000 & 1.0000 & 0.0000\end{bmatrix}$$
$$L_e = \begin{bmatrix}0.3162 & 1.0000 & 1.0000\end{bmatrix}$$
$$K_e = \begin{bmatrix}20.0000\\ 5.4031\\ 1.0000\end{bmatrix}$$

Bode magnitude plots after optimization

$$G_{zv}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \\ \frac{P}{s(1+PC)} & \frac{-PC}{s(1+PC)} \end{bmatrix}$$



Summary of LQG

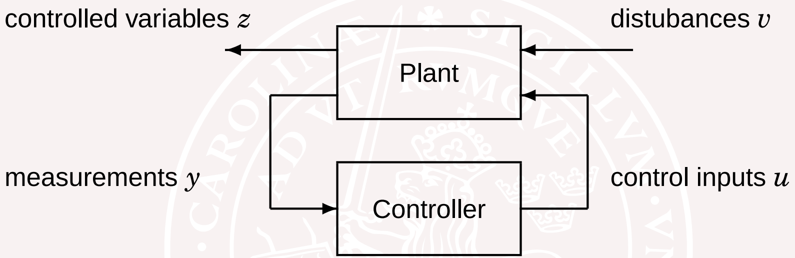
Advantages

- Works fine with multivariable models
- Observer structure ties to reality
- Always stabilizing
- Guaranteed robustness in state feeback case
- Well developed theory

Disadvantages

- High-order controllers
- Sometimes hard to choose weights

Alternative norms for optimization



LQG optimal control:

Minimize $$\int_{-\infty}^{\infty} G_{zv}(i\omega)G_{zv}(i\omega)^* d\omega$$

H_∞ optimal control:

Minimize $$\max_{\omega} \|G_{zv}(i\omega)\|$$

Linear Quadratic Game Problems

Notice that $\max_\omega \|G_{zv}(i\omega)\| \leq \gamma$ if and only if

$$|z|^2 - \gamma^2|v|^2 \leq 0$$

for all solutions to the system equations.

The H_∞ optimal control problem with $|z|^2 = x^T Q_1 x + u^T Q_2 u$ can be restated in terms of linear quadratic games of the form

$$\min_u \max_v (x^T Q_1 x + u^T Q_2 u - \gamma^2 |v|^2)$$

These can be solved using Riccati equations, just like LQG.

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L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

12. Internal Model Control, Youla parametrization
13. Synthesis by convex optimization
14. Controller simplification