FRTN10 Multivariable Control, Lecture 3

Anton Cervin

Automatic Control LTH, Lund University

Course Outline

L1-L5 Specifications, models and loop-shaping by hand

1. Introduction and system representations
2. Stability and robustness
3. Specifications and disturbance models
4. Control synthesis in frequency domain
5. Case study

L6-L8 Limitations on achievable performance

L9-L11 Controller optimization: Analytic approach

L12-L14 Controller optimization: Numerical approach

Lecture 3: Specifications and Disturbance Models

Continuing from lecture 2...

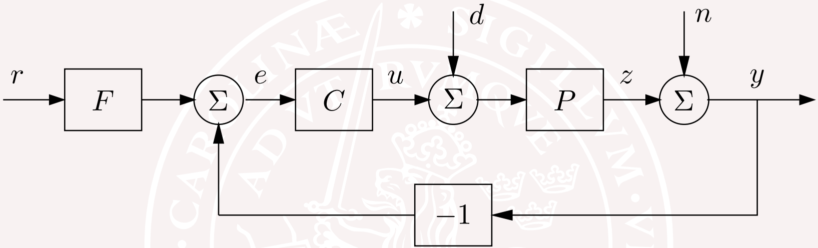
- Look at all transfer functions the closed-loop system! (Gang of Four / Gang of six)
- Scalings

New today

- Stochastic disturbances
- From transfer function to output spectrum
- From output spectrum to transfer function

[Glad & Ljung] Ch. 5.1–5.6, 6.1–6.3

A Basic Control System



Ingredients:

- Controller: feedback C, feedforward F
- Load disturbance d: Drives the system from desired state
- Measurement noise n: Corrupts information about z
- Process variable z should follow reference r

Design problem

Find a controller that

A: reduces the effect of load disturbances

B: does not inject too much measurement noise into the system

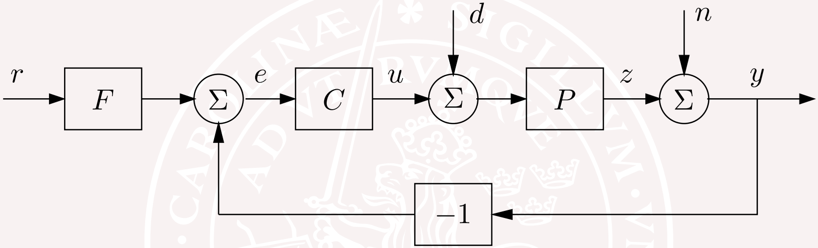
C: makes the closed loop insensitive to process variations

D: makes the output follow the setpoint

It is convenient to use a controller with **two degrees of freedom**, i.e. separate signal transmission from y to u and from r to u. This gives a nice separation of the design problem:

1. First design feedback compensator to deal with A, B, and C.
2. Then design feedforward compensator to deal with D.

Relations between signals



$$Z = \frac{P}{1+PC}D - \frac{PC}{1+PC}N + \frac{PCF}{1+PC}R$$

$$Y = \frac{P}{1+PC}D + \frac{1}{1+PC}N + \frac{PCF}{1+PC}R$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N + \frac{CF}{1+PC}R$$

The Gang of SIx

Six transfer functions are required to show the properties of a basic feedback loop. Four characterize the response to load disturbances and measurement noise.

$$\frac{PC}{1+PC} \qquad \frac{P}{1+PC}$$
$$\frac{C}{1+PC} \qquad \frac{1}{1+PC}$$

Two more are required to describe the response to set point changes.

$$\frac{PCF}{1+PC} \qquad \frac{CF}{1+PC}$$

Some Observations

- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four)
- The system with a controller having two degrees of freedom is characterized by *six* transfer function (The Gang of Six)
- To fully understand a system it is necessary to look at **all** transfer functions
- It may be strongly misleading to show properties of only one or a few transfer functions, for example the response of the output to command signals. This is a common error in the literature.
- The properties of the different transfer functions can be illustrated by their transient or frequency responses.

Some Observations

- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four)
- The system with a controller having two degrees of freedom is characterized by *six* transfer function (The Gang of Six)
- To fully understand a system it is necessary to look at **all** transfer functions
- It may be strongly misleading to show properties of only one or a few transfer functions, for example the response of the output to command signals. This is a common error in the literature.
- The properties of the different transfer functions can be illustrated by their transient or frequency responses.

Some Observations

- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four)
- The system with a controller having two degrees of freedom is characterized by *six* transfer function (The Gang of Six)
- To fully understand a system it is necessary to look at **all** transfer functions
- It may be strongly misleading to show properties of only one or a few transfer functions, for example the response of the output to command signals. This is a common error in the literature.
- The properties of the different transfer functions can be illustrated by their transient or frequency responses.

Some Observations

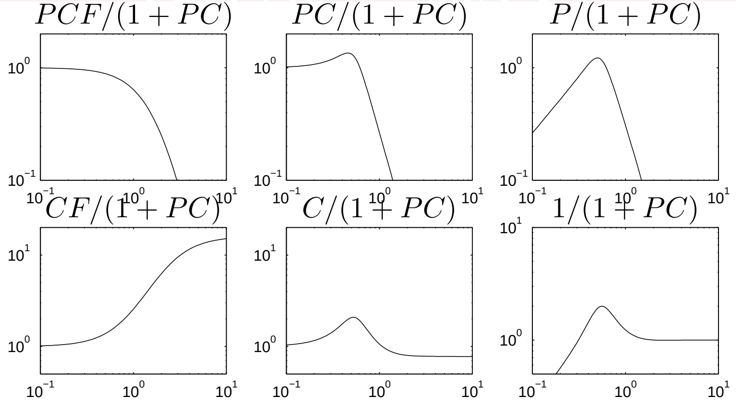
- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four)
- The system with a controller having two degrees of freedom is characterized by *six* transfer function (The Gang of Six)
- To fully understand a system it is necessary to look at **all** transfer functions
- It may be strongly misleading to show properties of only one or a few transfer functions, for example the response of the output to command signals. This is a common error in the literature.
- The properties of the different transfer functions can be illustrated by their transient or frequency responses.

Some Observations

- A system based on error feedback is characterized by *four* transfer functions (The Gang of Four)
- The system with a controller having two degrees of freedom is characterized by *six* transfer function (The Gang of Six)
- To fully understand a system it is necessary to look at **all** transfer functions
- It may be strongly misleading to show properties of only one or a few transfer functions, for example the response of the output to command signals. This is a common error in the literature.
- The properties of the different transfer functions can be illustrated by their transient or frequency responses.

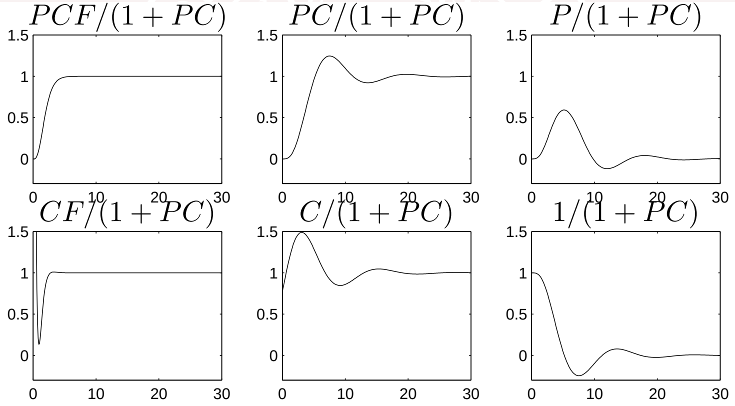
Amplitude Curves of Frequency Responses

Example: PI control with $K = 0.775$, $T_i = 2.05$ of $P(s) = (s+1)^{-4}$ with $G_{r\to y}(s) = (0.5s+1)^{-4}$



Step Responses

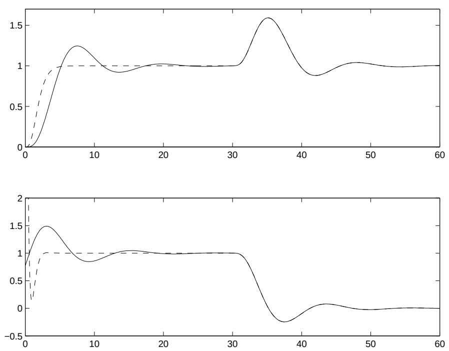
Example: PI control with $K = 0.775$, $T_i = 2.05$ of $P(s) = (s+1)^{-4}$ with $G_{r\to y}(s) = (0.5s+1)^{-4}$



Step Reponses—An Alternative

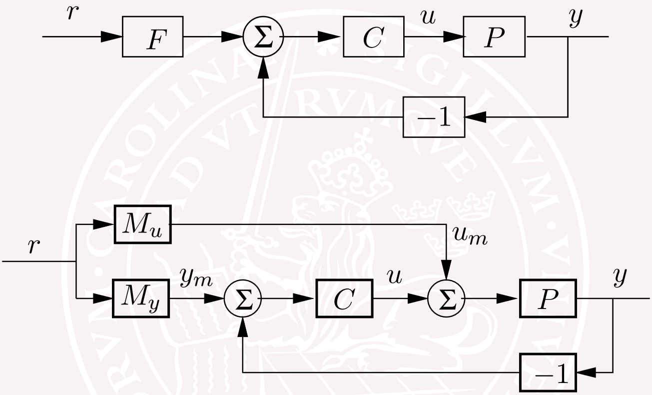
Show the responses in the output and the control signal to a step change in the reference signal for system with pure error feedback and with feedforward. Keep the reference signal constant and make a unit step in the process input.

(Upper:) Output response *(Lower:)* Control signal.



step responce load disturbance

Many Versions of 2DOF



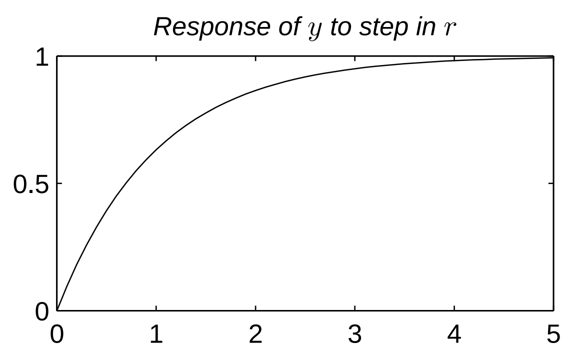
For linear systems all 2DOF configurations have the same properties.

For the systems above we have

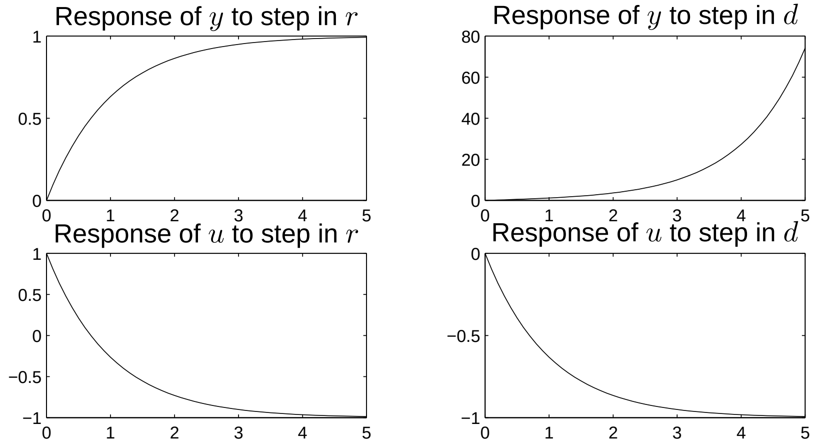
$$CF = M_u + CM_y$$

A Warning!

*Remember to always look at **all** responses when you are dealing with control systems. The step response below looks fine, but . . .*



Gang of Four



What is going on?

The System

Process $P(s) = \dfrac{1}{s-1}$

Controller $C(s) = \dfrac{s-1}{s}$

Response of y to reference r

$$\frac{Y(s)}{R(s)} = \frac{PC}{1+PC} = \frac{1}{s+1}$$

Response of y to step in disturbance d

$$\frac{Y(s)}{D(s)} = \frac{P}{1+PC} = \frac{s}{s^2-1} = \frac{s}{(s+1)(s-1)}$$

Scaling

The norms used to measure signal size can be very misleading if we are using states with very different magnitudes

Common to scale/normalize variables for state representations

$$x_i = x_i^p / d_i$$

where

- x_i^p corresponds to physical units
- d_i corresponds to (expected) max size of variable (absolute value).

Alternative: Use a weighed signal norm, e.g. $\|u\|_Q = \sqrt{\int_0^\infty u^T Q u \, dt}$, where Q is a positive semidefinite matrix

Lecture 3: Specifications and Disturbance Models

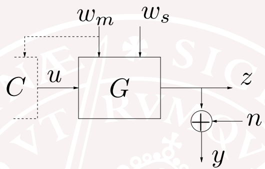
Continuing from lecture 2...

- Look at all transfer functions the closed-loop system! (Gang of Four / Gang of six)
- Scalings

New today

- Stochastic disturbances
- From transfer function to output spectrum
- From output spectrum to transfer function

Disturbances cont.



Load disturbances

- disturbances which really affect the system
  - w_m measurable — use e.g., in feedforward compensation
  - w_s non-measurable — controller need to suppress these

Measurement disturbances n

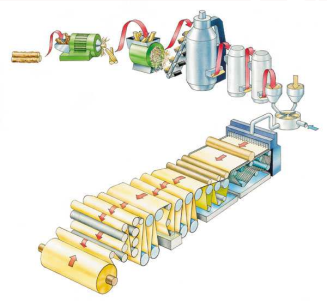
- Controller should not be "fooled" by measurement disturbances

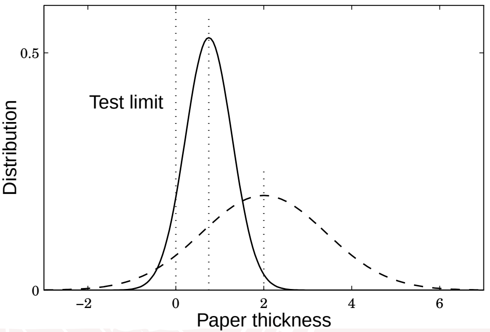
Common case: $z = \mathcal{S}(u, w_m, w_s), \qquad y = z + n$ where

z is the control objective, $\qquad y$ is the measured output

Motivation

Example: Paper thickness — want to keep down variation in output!





All paper production below the test limit is wasted.

Good control allows for lower setpoint with the same waste. The average thickness is lower, which saves significant costs.

Motivation cont'd - LQG control

For a system with process noise w and measurement noise v, where v is white noise with intensity R_1 and w is white noise with intensity R_2, find a feedback law from y to u that solves the following optimization problem:

Minimize $\mathbf{E}\left(x^T Q_1 x + 2x^T Q_{12} u + u^T Q_2 u\right)$

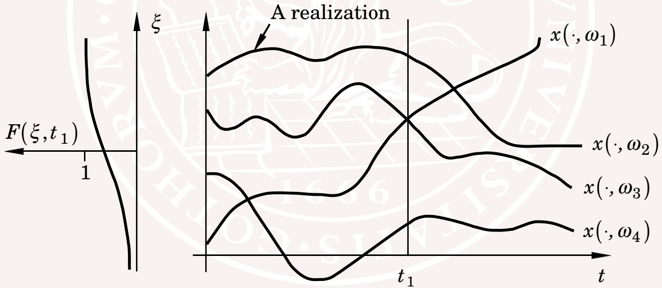
subject to

$$\dot{x} = Ax + Bu + w$$
$$y = Cx + Du + v$$

Stochastic processes

A **stochastic process** (random process) is a family of stochastic variables $\{x(t), t \in T\}$
A function of two variables $x(t, \omega)$

Fixed $\omega = \omega_0$ gives a time function $x(\cdot, \omega_0)$ (realization)
Fixed $t = t_1$ gives a random variable $x(t_1, \cdot)$



Zero mean stationary stochastic processes

The distribution is independent of t

Mean-value function

$$\mathbf{E}x(t) \equiv 0$$

Covariance function

$$r_{xx}(\tau) = \mathbf{E}x(t+\tau)x(t)^T$$

Cross-covariance function

$$r_{xy}(\tau) = \mathbf{E}x(t+\tau)y(t)^T$$

A zero mean Gaussian process x is completely determined by its covariance function.

Spectral density

Define the *spectral density* as the Fourier transform of the covariance function

$$\Phi_{xy}(\omega) := \int_{-\infty}^{\infty} r_{xy}(t) e^{-it\omega} dt$$

Then, by inverse Fourier transform

$$r_{xy}(t) = \int_{-\infty}^{\infty} e^{it\omega} \Phi_{xy}(\omega)\, d\omega$$

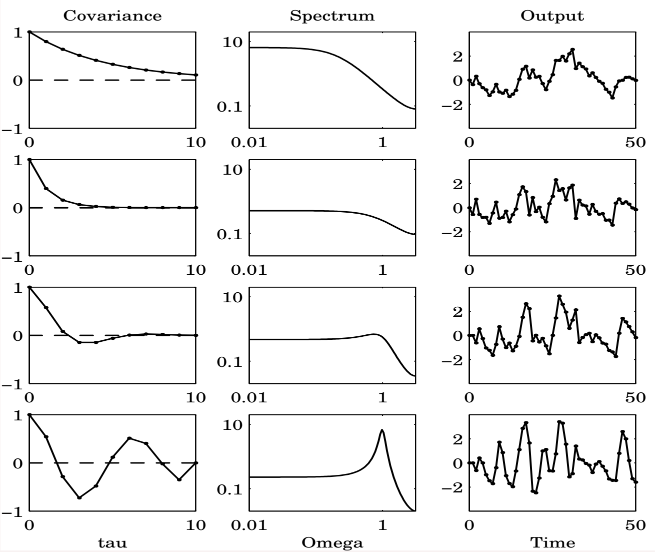
In particular

$$\mathbf{E} x(t) x^T(t) = r_{xx}(0) = \int_{-\infty}^{\infty} \Phi_{xx}(\omega)\, d\omega$$

White noise with *intensity* R means a process e such that

$$\Phi_e(\omega) = R \quad \text{for all frequencies } \omega$$

Covariance, spectral density, and realization



Error correction: The spectra should be divided by 2π

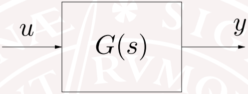
Two Problems

1. Determine covariance function and spectral density of y when white noise u is filtered through a linear system $G(s)$ or

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

2. Conversely, find $G(s)$ or state-space matrices A, B and C to give y a desired spectral density.

Spectral density and transfer functions



Assume that u has spectral density $\Phi_u(\omega)$ and y is obtained by filtering u with the transfer function $G(i\omega)$.

Then y gets the spectral density

$$\Phi_y(\omega) = G(i\omega)\Phi_u(\omega)G(i\omega)^*$$

and the cross-spectral density becomes

$$\Phi_{yu}(\omega) = G(i\omega)\Phi_u(\omega)$$

.

Linear system with white noise input

Consider the linear system

$$\dot{x} = Ax + Bv, \qquad \Phi_v(\omega) = R$$

The transfer function from v to x is

$$G(s) = (sI - A)^{-1}B$$

and the spectrum for x will be

$$\Phi_x(\omega) = (i\omega I - A)^{-1} B R \underbrace{B^*(-i\omega I - A)^{-T}}_{G(i\omega)^*}$$

Covariance matrix for state x:

$$\Pi_x = R_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega) d\omega$$

Calculating the state covariance matrix

Theorem [G&L 5.3]

If all eigenvalues of A are strictly in the left half plane then there exists a unique matrix $\Pi_x = \Pi_x^T > 0$ which is the solution to the Lyuapunov equation

$$A\Pi_x + \Pi_x A^T + BRB^T = 0$$

Example: Consider the system

$$\dot{x} = Ax + Bv \quad = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

where v is white noise with intensity 1.

What is the covariance of x?

First check the eigenvalues of A : $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \in LHP$. OK!

Solve the Lyapunov equation $A\Pi_x + \Pi_x A^T + BRB^T = 0_{2,2}$.

Example: Consider the system

$$\dot{x} = Ax + Bv \quad = \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} v$$

where v is white noise with intensity 1.

What is the covariance of x?

First check the eigenvalues of A : $\lambda = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2} \in LHP$. OK!

Solve the Lyapunov equation $A\Pi_x + \Pi_x A^T + BRB^T = 0_{2,2}$.

Example cont'd

$$A\Pi_x + \Pi_x A^T + BRB^T = 0_{2\times 2}$$

Find Π_x:

$$\begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} + \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2(-\Pi_{11} + 2\Pi_{12}) + 1 & -\Pi_{12} + 2\Pi_{22} - \Pi_{11} \\ -\Pi_{12} + 2\Pi_{22} - \Pi_{11} & -2\Pi_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving for Π_{11}, Π_{12} and Π_{22} gives

$$\Longrightarrow \Pi_x = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} > 0$$

Matlab: `lyap([-1 2; -1 0],[1 ; 0]*[1 0])`

Example cont'd

$$A\Pi_x + \Pi_x A^T + BRB^T = 0_{2\times 2}$$

Find Π_x:

$$\begin{bmatrix}-1 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12} & \Pi_{22}\end{bmatrix} + \begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12} & \Pi_{22}\end{bmatrix}\begin{bmatrix}-1 & -1\\ 2 & 0\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix} =$$

$$= \begin{bmatrix}2(-\Pi_{11} + 2\Pi_{12}) + 1 & -\Pi_{12} + 2\Pi_{22} - \Pi_{11}\\ -\Pi_{12} + 2\Pi_{22} - \Pi_{11} & -2\Pi_{12}\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Solving for Π_{11}, Π_{12} and Π_{22} gives

$$\Longrightarrow \Pi_x = \begin{bmatrix}\Pi_{11} & \Pi_{12}\\ \Pi_{12} & \Pi_{22}\end{bmatrix} = \begin{bmatrix}1/2 & 0\\ 0 & 1/4\end{bmatrix} > 0$$

Matlab: `lyap([-1 2; -1 0],[1 ; 0]*[1 0])`

Example cont'd

$$A\Pi_x + \Pi_x A^T + BRB^T = 0_{2\times 2}$$

Find Π_x:

$$\begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} + \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 2(-\Pi_{11} + 2\Pi_{12}) + 1 & -\Pi_{12} + 2\Pi_{22} - \Pi_{11} \\ -\Pi_{12} + 2\Pi_{22} - \Pi_{11} & -2\Pi_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Solving for Π_{11}, Π_{12} and Π_{22} gives

$$\Longrightarrow \Pi_x = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12} & \Pi_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix} > 0$$

Matlab: `lyap([-1 2; -1 0],[1 ; 0]*[1 0])`

Spectral Factorization — Example

Find a filter $G(s)$ such that a process y generated by filtering unit intensity white noise through G will give

$$\Phi_y(\omega) = \frac{\omega^2 + 4}{\omega^4 + 10\omega^2 + 9},$$

Solution. We have

$$\Phi_y(\omega) = \frac{\omega^2 + 4}{(\omega^2 + 1)(\omega^2 + 9)} = \left| \frac{i\omega + 2}{(i\omega + 1)(i\omega + 3)} \right|^2$$

so $G(s) = \frac{s+2}{(s+1)(s+3)}$ works. So does $G(s) = \frac{s-2}{(s+1)(s+3)}$.

Summary of today's most important concepts

- Gang of four / gang of six
- Scalings
- Stochastic disturbances, described by covariance functions or spectral densities
- White noise
- Translation from generating transfer function to output spectrum
- Translation from output spectrum to generating transfer function (spectral factorization)