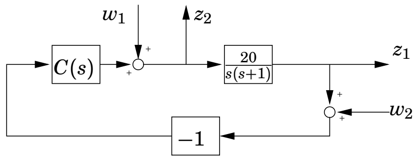
Lecture 14: Controller simplification

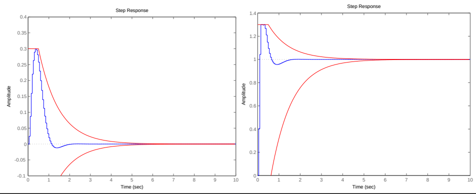
- **Model reduction by balanced truncation**
  - Application to controller simplification
  - Frequency weighted balanced truncation

Model reduction by balanced truncation is described in Glad/Ljung, section 3.6.

Example — DC-motor



We previously minimized $\int_{-\infty}^{\infty} G_{zw}(i\omega)G_{zw}(i\omega)^* d\omega$ subject to step response bounds on the transfer functions from w_1 and w_2 to z_1:

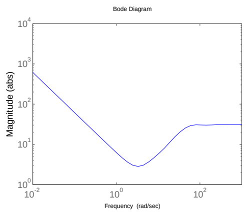


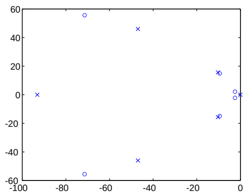
The optimized controller has high order

Recall that $C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1}Q(s)$ with $Q(s) = \sum_{k=0}^{N} Q_k\phi_k(s)$.

Hence the controller order will grow with the number of basis functions ϕ_k and their complexity.

However, in the DC-servo example, both the Bode diagram and pole-zero diagram of the controller indicate that cancellations can be done to simplify the controller.





Controllability and Observability

The controllability Gramian $S = \int_0^\infty e^{At}BB^Te^{A^Tt}dt$ can be computed by solving the linear system of equations

$$AS + SA^T + BB^T = 0$$

The observability Gramian $O = \int_0^\infty e^{A^Tt}C^TCe^{At}dt$ can be computed by solving the linear system of equations

$$A^TO + OA + C^TC = 0$$

We want to remove states that are either poorly controllable or poorly observable.

Gramians, looking back

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Impulse response from zero intial condition: $u_i(t) = \delta(t)$, $x(0) = 0$

$$\begin{aligned} x_i(t) &= e^{At}B_i \\ X(t) &= \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} = e^{At}B \\ S_x &\hat{=} \int_0^\infty X(t)X^T(t)\,dt = \int_0^\infty e^{At}BB^Te^{A^Tt}\,dt \end{aligned}$$

Output from $u \equiv 0$ (only initial state $x(0) = x_0$)

$$y(t) = Cx(t) = Ce^{At}x_0$$

$$\int_0^\infty y(t)^Ty(t)dt = \int_0^\infty x_0^Te^{A^Tt}C^TCAtx_0dt \quad \hat{=} \quad x_0^TO_xx_0$$

Balanced Realizations

For a stable system (A, B, C) with gramians S_x and O_x, the variable transformation $\xi = Tx$ gives the new state matrices $\widehat{A} = TAT^{-1}$, $\widehat{B} = TB$, $\widehat{C} = CT^{-1}$ and the new gramians

$$S_\xi = \int_0^\infty e^{\widehat{A}t}\widehat{B}\widehat{B}^Te^{\widehat{A}^Tt}dt = \int_0^\infty Te^{At}BB^Te^{A^Tt}T^Tdt = TS_xT^T$$

$$O_\xi = \int_0^\infty e^{\widehat{A}^Tt}\widehat{C}^T\widehat{C}e^{\widehat{A}t}dt = \int_0^\infty T^{-T}e^{A^Tt}C^TCe^{A^Tt}T^{-1}dt = T^{-T}O_xT^{-1}$$

A particular choice of T gives $S_\xi = O_\xi = \underbrace{\begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}}_{\Sigma}$

The corresponding realization

$$\begin{cases} \dot{\xi} = \widehat{A}\xi + \widehat{B}u \\ y = \widehat{C}x \end{cases}$$

is called a *balanced realization*.

Hankel singular values

Notice that

$$\begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_n^2 \end{bmatrix} = \underbrace{(TS_xT^T)}_{\Sigma}\underbrace{(T^{-T}O_xT^{-1})}_{\Sigma} = TS_xO_xT^{-1}$$

so the diagonal elements are the eigenvalues of S_xO_x, independently of coordinate system. The numbers $\sigma_1, \ldots, \sigma_n$ are called the *Hankel singular values* of the system.

A small Hankel singular value corresponds to a state that is both weakly controllable and weakly observable. Hence, it can be truncated without much effect on the input-output behavior.

Model reduction by balanced truncation

Consider a balanced realization

$$\begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} + Du$$

with the lower part of the gramian being $\Sigma_2 = \begin{bmatrix} \sigma_{r+1} & & 0 \\ & \ddots & \\ 0 & & \sigma_n \end{bmatrix}$.

Replacing the second state equation by $\dot{\xi}_2 = 0$ gives the relation $0 = A_{21}\xi_1 + A_{22}\xi_2 + B_2u$. The reduced system

$$\begin{cases} \dot{\xi}_1 = (A_{11} - A_{12}A_{22}^{-1}A_{21})\xi_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_r = (C_1 - C_2A_{22}^{-1}A_{21})\xi_1 + (D - C_2A_{22}^{-1}B_2)u \end{cases}$$

satisfies the error bound

$$\frac{\|y - y_r\|_2}{\|u\|_2} \le 2\sigma_{r+1} + \cdots + 2\sigma_n$$

Example 1

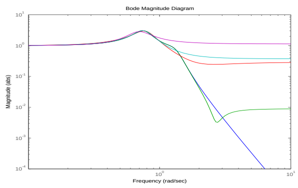
Original system: $\dfrac{1-s}{s^6+3s^5+5s^4+7s^3+5s^2+3s+1}$

Hankel singular values:

```
Sigma = [1.9837   1.9184   0.7512   0.3292   0.1478   0.0045]
```

Reduced system:

```
 0.3717 s^3 - 0.9682 s^2 + 1.14 s - 0.5185
 -----------------------------------------
     s^3 + 1.136 s^2 + 0.825 s + 0.5185
```



Example 2 — Heat Exchanger

$$V_C \frac{dT_C}{dt} = f_C(T_{C_i} - T_C) + \beta(T_H - T_C) \qquad \text{(cold side)}$$
$$V_H \frac{dT_H}{dt} = f_H(T_{H_i} - T_H) - \beta(T_H - T_C) \qquad \text{(hot side)}$$

$u_1 = T_{C_i}$ is the in-flow temperature on the cold side
$x_1 = T_C$ is the out-flow temperature on the cold side
$u_2 = T_{H_i}$ is the in-flow temperature on the hot side
$x_2 = T_H$ is the out-flow temperature on the hot side

Numerical values:

$$\dot{x} = \begin{bmatrix} -0.21 & 0.2 \\ 0.2 & -0.21 \end{bmatrix} x + \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix} u$$
$$y = x$$

Example 2 — Heat Exchanger

A state transformation $\xi_1 = -7.07(x_1 + x_2)$, $\xi_2 = 7.07(x_1 - x_2)$ gives the balanced realization

$$\dot{\xi} = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.41 \end{bmatrix} \xi + 0.0707 \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix} u$$
$$y = 0.0707 \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \xi$$

the common controllability/observability matrix

$$S_\xi = O_\xi = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.0122 \end{bmatrix}$$

and the reduced model

$$\dot{\xi}_1 = -0.01\xi_1 - 0.0707 \begin{bmatrix} 1 & 1 \end{bmatrix} u$$
$$y = -0.0707 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xi_1 + 0.0122 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} u$$

DC-servo again

To simplify the controller, we would like to remove states that have little influence on the input-output relationship, i.e. states that are poorly controllable or poorly observable.

For this, we will compute the controllability gramian and the observability gramian. However, these are defined only for stable systems. Hence the integrator needs to be treated separately:

$$C_{\text{opt}}(s) = C_{\text{stab}}(s) - \frac{6.17}{s}$$

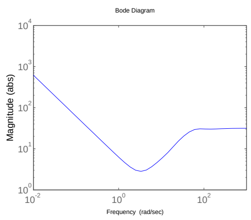
For $C_{\text{stab}}(s)$ the gramians have eigenvalues

```
eig(ConGram) = [0.0933  0.2972   0.9417   5.9373  50.0472]
eig(ObsGram) = [0.0291  0.0913   0.2964   1.8811  17.6379]
```

Three out of five states are poorly controllable and three are weakly observable. This can be used for reduction!

Reducing the DC-servo Controller

Recall the Bode plot of the optimized controller $C_{\text{opt}}(s)$:



The Hankel singular values of $C_{\text{stab}}(s) = C_{\text{opt}}(s) + \frac{6.17}{s}$ are

```
Sigma = [16.0768  2.2306  0.7023  0.1994  0.0896]
```

How many states need to be kept in $C_{\text{stab}}(s)$?
What kind of controller remains?

Are all frequencies equally important?

The error bound

$$\max_\omega |G(i\omega) - G_r(i\omega)| = \sup_u \frac{\|y - y_r\|_2}{\|u\|_2} \le 2\sigma_{r+1} + \cdots + 2\sigma_n$$

emphasizes all frequencies equally, but comparing a controller $C(s)$ with a reduced controller $C_r(s)$ in closed loop operation gives

$$|P(I+CP)^{-1}C - P(I+C_rP)^{-1}C_r| \approx |P(I+CP)^{-1}(C - C_r)|$$

Hence it is interesting to minimize the frequency weighted error

$$\max_\omega \Big| W(i\omega)[C(i\omega) - C_r(i\omega)] \Big|$$

where $W(i\omega) = P(i\omega)(I + C(i\omega)P(i\omega))^{-1}$.

Frequency weighted balanced truncation

For model reduction with the aim to minimize

$$\max_\omega \Big\| W_o(i\omega)[G(i\omega) - G_r(i\omega)]W_i(i\omega) \Big\|$$

where

$W_i(s) = C_i(sI - A_i)^{-1}B_i + D_i \quad G(s) = C(sI - A)^{-1}B + D \quad W_o(s) = C_o(sI - A_o)^{-1}B_o + D_o$

find extended gramians by solving

$$\begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix} \begin{bmatrix} S & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} + \begin{bmatrix} S & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix} \begin{bmatrix} A & BC_i \\ 0 & A_i \end{bmatrix}^T + \begin{bmatrix} BD_i \\ B_i \end{bmatrix} \begin{bmatrix} BD_i \\ B_i \end{bmatrix}^T = 0$$
$$\begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix}^T \begin{bmatrix} O & O_{12} \\ O_{12}^T & O_{22} \end{bmatrix} + \begin{bmatrix} O & O_{12} \\ O_{12}^T & O_{22} \end{bmatrix} \begin{bmatrix} A & 0 \\ B_oC & A_o \end{bmatrix} + \begin{bmatrix} C^TD_o^T \\ D_o^T \end{bmatrix} \begin{bmatrix} D_oC & D_o \end{bmatrix} = 0$$

then change coordinates to make S and O equal and diagonal before truncating the realization of $G(s)$ to get $G_r(s)$ as before.

Summary

- Low order controllers could be desirable to meet constraints on speed and memory.
- Balanced realizations can reveal less important states
- Good theoretical error bounds
- Frequency weighting essential for closed loop performance
- Reduction of unstable controllers not treated here