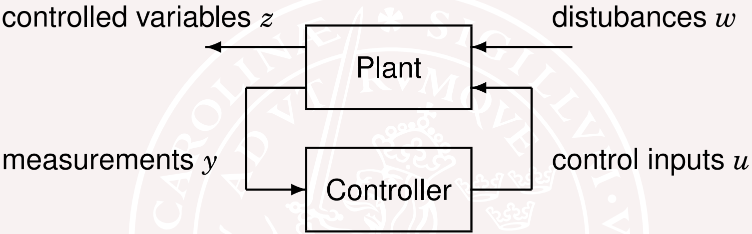
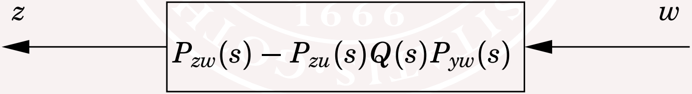
# The $Q$-parametrization (Youla)




**Idea for lecture 12-14:**
The choice of controller generally corresponds to finding $Q(s)$, to get desirable properties of the map from $w$ to $z$:



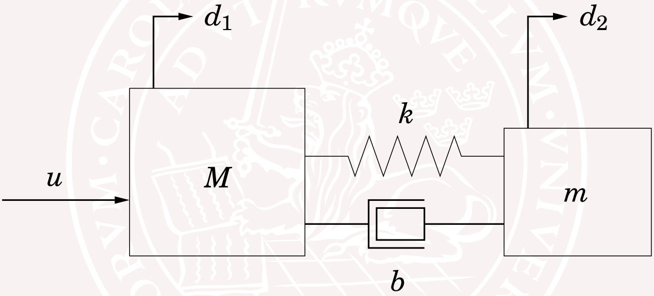

Once $Q(s)$ is determined, a corresponding controller is derived.

# Lecture 13: Synthesis by Convex Optimization

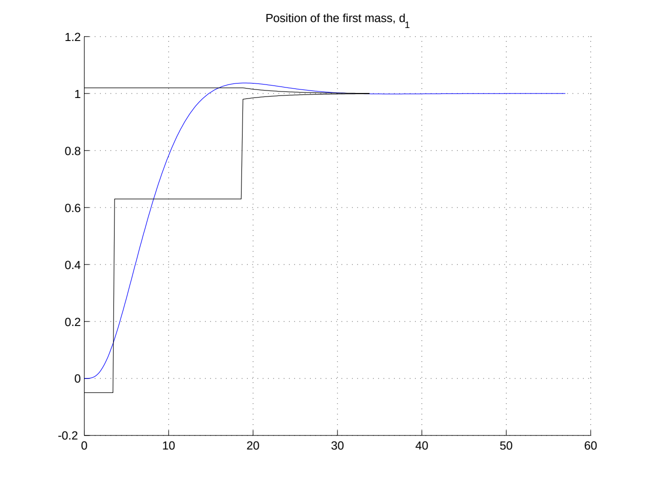
- **Example: Spring-mass system**
- Introduction to convex optimization
- Controller optimization using Youla parametrization
- Examples revisited

Most of this lecture is based on source material from Boyd, Vandenberghe and coauthors. See
http://www.control.lth.se/Education/EngineeringProgram/FRTN10.html

# Example: Spring-mass System

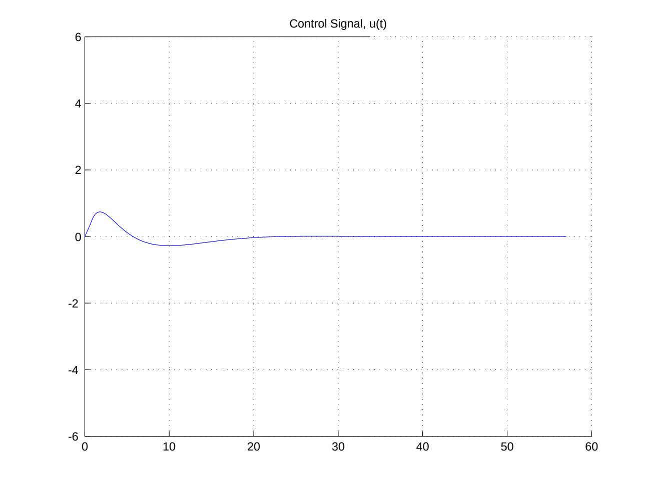


# Lecture 13: Synthesis by Convex Optimization



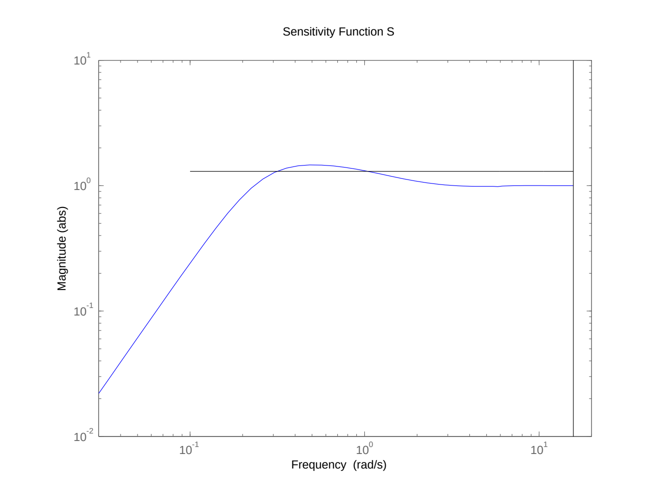

The step response is not within its upper and lower bounds.

# Lecture 13: Synthesis by Convex Optimization




The step input stays within its amplitude bound $|u(t)| \leq 6$.

# Lecture 13: Synthesis by Convex Optimization




The sensitivity does not satisfy the magnitude bound $|S| \leq 1.3$

# Lecture 13: Synthesis by Convex Optimization

- ◦ Example: Spring-mass system
- • **Introduction to convex optimization**
- ◦ Controller optimization using Youla parametrization
- ◦ Examples revisited

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# Least-squares

$$\text{minimize} \quad \|Ax - b\|_2^2$$

**solving least-squares problems**

- analytical solution: $x^\star = (A^T A)^{-1} A^T b$
- reliable and efficient algorithms and software
- computation time proportional to $n^2 k$ ($A \in \mathbf{R}^{k \times n}$); less if structured
- a mature technology

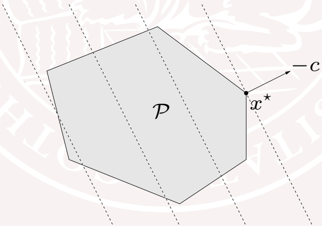
**using least-squares**

- least-squares problems are easy to recognize
- a few standard techniques increase flexibility (*e.g.*, including weights, adding regularization terms)

# Linear program (LP)

$$\begin{array}{ll} \text{minimize} & c^T x + d \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- convex problem with affine objective and constraint functions
- feasible set is a polyhedron



# Linear programming

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & a_i^T x \leq b_i, \quad i = 1, \ldots, m \end{array}$$

**solving linear programs**

- no analytical formula for solution
- reliable and efficient algorithms and software
- computation time proportional to $n^2 m$ if $m \geq n$; less with structure
- a mature technology

**using linear programming**

- not as easy to recognize as least-squares problems
- a few standard tricks used to convert problems into linear programs ($e.g.$, problems involving $\ell_1$- or $\ell_\infty$-norms, piecewise-linear functions)

# Convex optimization problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i, \quad i = 1, \dots, m \end{array}$$

- objective and constraint functions are convex:

$$f_i(\alpha x + \beta y) \le \alpha f_i(x) + \beta f_i(y)$$

  if $\alpha + \beta = 1$, $\alpha \ge 0$, $\beta \ge 0$

- includes least-squares problems and linear programs as special cases

**solving convex optimization problems**

- no analytical solution
- reliable and efficient algorithms
- computation time (roughly) proportional to $\max\{n^3, n^2m, F\}$, where $F$ is cost of evaluating $f_i$'s and their first and second derivatives
- almost a technology

**using convex optimization**

- often difficult to recognize
- many tricks for transforming problems into convex form
- surprisingly many problems can be solved via convex optimization

# Brief history of convex optimization

**theory (convex analysis)**: ca1900–1970

**algorithms**

- 1947: simplex algorithm for linear programming (Dantzig)
- 1960s: early interior-point methods (Fiacco & McCormick, Dikin, . . . )
- 1970s: ellipsoid method and other subgradient methods
- 1980s: polynomial-time interior-point methods for linear programming (Karmarkar 1984)
- late 1980s–now: polynomial-time interior-point methods for nonlinear convex optimization (Nesterov & Nemirovski 1994)

**applications**

- before 1990: mostly in operations research; few in engineering
- since 1990: many new applications in engineering (control, signal processing, communications, circuit design, . . . ); new problem classes (semidefinite and second-order cone programming, robust optimization)

## Examples on $\mathbf{R}$

convex:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- exponential: $e^{ax}$, for any $a \in \mathbf{R}$
- powers: $x^\alpha$ on $\mathbf{R}_{++}$, for $\alpha \geq 1$ or $\alpha \leq 0$
- powers of absolute value: $|x|^p$ on $\mathbf{R}$, for $p \geq 1$
- negative entropy: $x \log x$ on $\mathbf{R}_{++}$

concave:

- affine: $ax + b$ on $\mathbf{R}$, for any $a, b \in \mathbf{R}$
- powers: $x^\alpha$ on $\mathbf{R}_{++}$, for $0 \leq \alpha \leq 1$
- logarithm: $\log x$ on $\mathbf{R}_{++}$

# Examples on $\mathbf{R}^n$ and $\mathbf{R}^{m\times n}$

affine functions are convex and concave; all norms are convex

**examples on $\mathbf{R}^n$**

- affine function $f(x) = a^T x + b$
- norms: $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ for $p \geq 1$; $\|x\|_\infty = \max_k |x_k|$

**examples on $\mathbf{R}^{m\times n}$** ($m \times n$ matrices)

- affine function

$$f(X) = \mathbf{tr}(A^T X) + b = \sum_{i=1}^m \sum_{j=1}^n A_{ij} X_{ij} + b$$

- spectral (maximum singular value) norm

$$f(X) = \|X\|_2 = \sigma_{\max}(X) = (\lambda_{\max}(X^T X))^{1/2}$$

# Convex optimization problem

**standard form convex optimization problem**

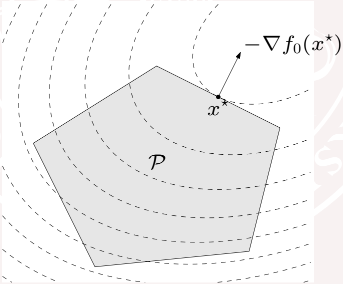
$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0, \quad i = 1, \ldots, m \\ & a_i^T x = b_i, \quad i = 1, \ldots, p \end{array}$$

- $f_0$, $f_1$, . . . , $f_m$ are convex; equality constraints are affine
- problem is *quasiconvex* if $f_0$ is quasiconvex (and $f_1$, . . . , $f_m$ convex)

## Quadratic program (QP)

$$\begin{array}{ll} \text{minimize} & (1/2)x^TPx + q^Tx + r \\ \text{subject to} & Gx \preceq h \\ & Ax = b \end{array}$$

- $P \in \mathbf{S}_+^n$, so objective is convex quadratic
- minimize a convex quadratic function over a polyhedron



# Second-order cone programming

$$\begin{array}{ll} \text{minimize} & f^T x \\ \text{subject to} & \|A_i x + b_i\|_2 \le c_i^T x + d_i, \quad i = 1, \ldots, m \\ & Fx = g \end{array}$$

$(A_i \in \mathbf{R}^{n_i \times n},\ F \in \mathbf{R}^{p \times n})$

# Semidefinite program (SDP)

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & x_1 F_1 + x_2 F_2 + \cdots + x_n F_n + G \preceq 0 \\ & Ax = b \end{array}$$

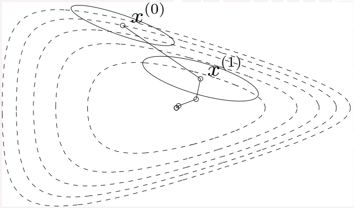
with $F_i$, $G \in \mathbf{S}^k$

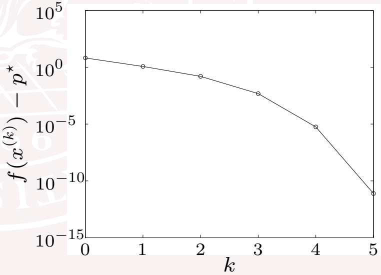
- inequality constraint is called linear matrix inequality (LMI)

# Newton's method

**given** a starting point $x \in \mathbf{dom}\, f$, tolerance $\epsilon > 0$.
**repeat**

1. *Compute the Newton step and decrement.*
   $$\Delta x_{\rm nt} := -\nabla^2 f(x)^{-1}\nabla f(x); \quad \lambda^2 := \nabla f(x)^T \nabla^2 f(x)^{-1}\nabla f(x).$$
2. *Stopping criterion.* **quit** if $\lambda^2/2 \le \epsilon$.
3. *Line search.* Choose step size $t$ by backtracking line search.
4. *Update.* $x := x + t\Delta x_{\rm nt}$.



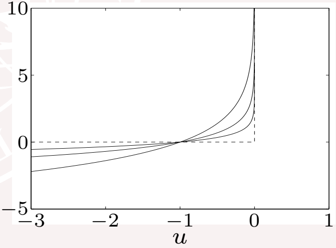
# Barrier method for constrained minimization

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad 1 = 1, \ldots, m \\ & Ax = b \end{array}$$

**approximation via logarithmic barrier**

$$\begin{array}{ll} \text{minimize} & f_0(x) - (1/t) \sum_{i=1}^{m} \log(-f_i(x)) \\ \text{subject to} & Ax = b \end{array}$$

- an equality constrained problem
- for $t > 0$, $-(1/t)\log(-u)$ is a smooth approximation of $I_-$
- approximation improves as $t \to \infty$



# Outline

- Example: Spring-mass system
- Introduction to convex optimization
- **Controller optimization using Youla parametrization**
- Examples revisited

# Scheme for numerical optimization of $Q$

Given some fixed set of basis function $\phi_0(s), \dots, \phi_N(s)$, we will search numerically for matrices $Q_0, \dots, Q_N$ such that the closed loop transfer matrix $G_{zw}(s)$ satisfies given specifications when

$$G_{zw}(s) = P_{zw}(s) - P_{zu}(s)Q(s)P_{yw}(s) \qquad \text{and } Q(s) = \sum_{k=0}^{N} Q_k \phi_k(s)$$

Once $Q(s)$ has been determined, we will recover the desired controller from the formula

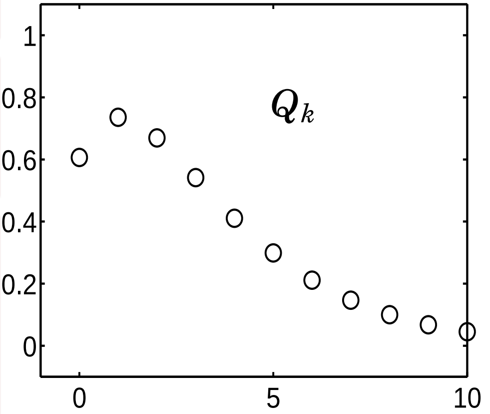
$$C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1} Q(s)$$

It is possible to choose the sequence $\phi_0(s), \phi_1(s), \phi_2(s), \dots$ such that every stable $Q$ can be approximated arbitrarily well. Hence, in principle, every convex control design problem can be solved this way.

But, what specifications give a convex design problem?

# Pulse response parameterization

We will use an intuitively simple parametrization of $Q(s)$ where each parameter $Q_k$ represents a point on the corresponding impulse response in time domain.

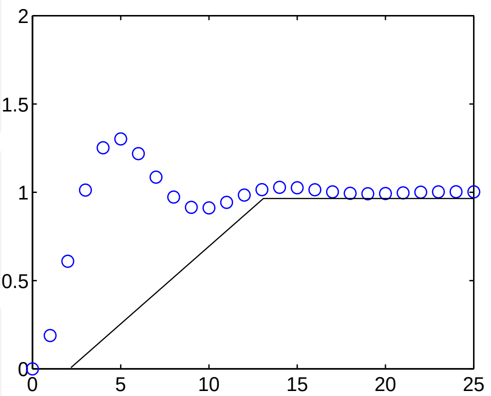


# Mini-problem

Which specifications are convex constraints on $Q_k$?

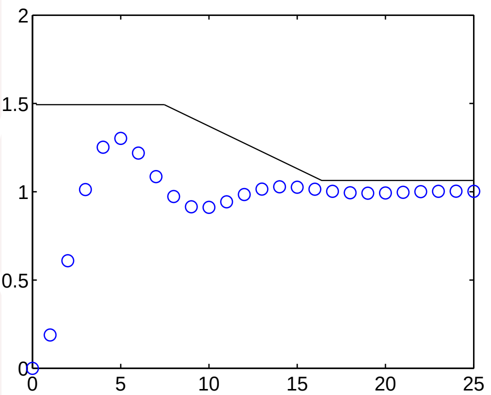
1. Stability of the closed loop system
2. Lower bound on step response from $w_i$ to $z_j$ at time $t_i$
3. Upper bound on step response from $w_i$ to $z_j$ at time $t_i$
4. Lower bound on Bode amplitude from $w_i$ to $z_j$ at frequency $\omega_i$
5. Upper bound on Bode amplitude from $w_i$ to $z_j$ at frequency $\omega_i$
6. Interval bound on Bode phase from $w_i$ to $z_j$ at frequency $\omega_i$

# Lower bound on step response



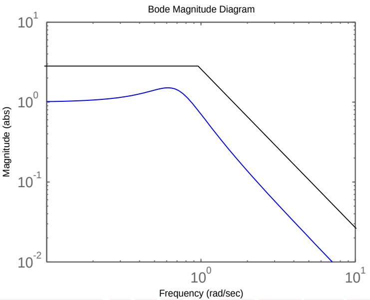

The step response depends linearly on $Q_k$, so every time $t_k$ with a lower bound gives a linear constraint.

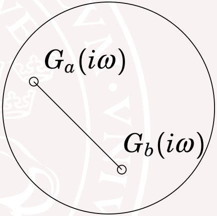
# Upper bound on step response




Every time $t_k$ with an upper bound also gives a linear constraint.

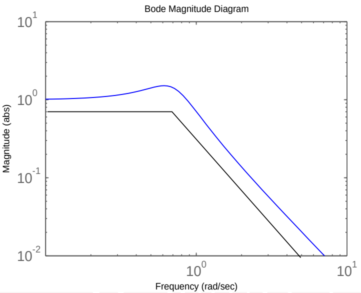
# Upper bound on Bode amplitude

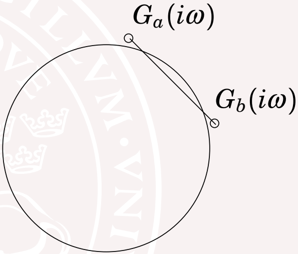



An amplitude bound $|G(i\omega_i)| < c$ is a quadratic constraint.

# Lower bound on Bode amplitude




An lower bound $|G(i\omega_i)|$ is a *non-convex* quadratic constraint. This should be avoided in optimization.

# Synthesis by convex optimization

A general control synthesis problem can be stated as a convex optimization problem in the variables $Q_0, \dots, Q_m$. The problem has a quadratic objective, with linear and quadratic constraints:

$$\text{Minimize} \quad \int_{-\infty}^{\infty} |P_{zw}(i\omega) + P_{zu}(i\omega) \overbrace{\sum_k Q_k \phi_k(i\omega)}^{Q(i\omega)} P_{yw}(i\omega)|^2 d\omega \quad \Big\} \text{ quadratc objective}$$

subject to

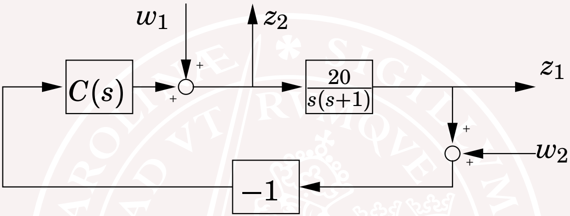
- step response $w_i \to z_j$ is smaller than $f_{ijk}$ at time $t_k$
- step response $w_i \to z_j$ is bigger than $g_{ijk}$ at time $t_k$ } linear constraints
- Bode magnitude $w_i \to z_j$ is smaller than $h_{ijk}$ at $\omega_k$ } quadratic constraints

Once the variables $Q_0, \dots, Q_m$ have been optimized, the controller is obtained as $C(s) = \left[I - Q(s)P_{yu}(s)\right]^{-1} Q(s)$

# Outline

- ◦ Example: Spring-mass system
- ◦ Introduction to convex optimization
- ◦ Controller optimization using Youla parametrization
- • **Examples revisited**

# Example — DC-motor




The transfer matrix from $(w_1, w_2)$ to $(z_1, z_2)$ is

$$G_{zw}(s) = \begin{bmatrix} \frac{P}{1+PC} & \frac{-PC}{1+PC} \\ \frac{1}{1+PC} & \frac{-C}{1+PC} \end{bmatrix}$$

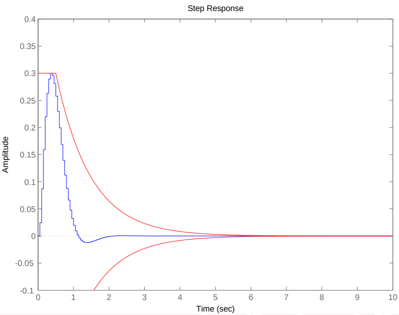
with $P(s) = \frac{20}{s(s+1)}$. We will choose $C(s)$ to minimize

$$\text{trace} \int_{-\infty}^{\infty} G_{zw}(i\omega) G_{zw}(i\omega)^* d\omega$$

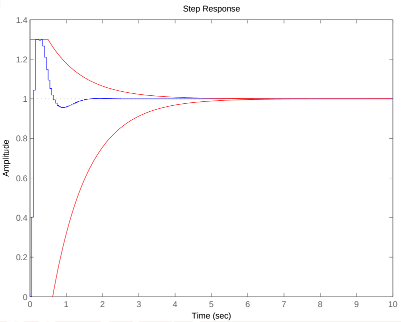
subject time-domain bounds.

# DC-servo with time domain bounds

Input step disturbance



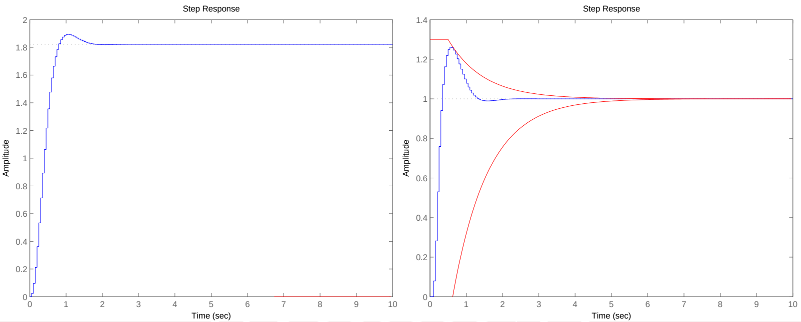

Reference step




What if we remove the upper bound on the response to input disturbances ?

# DC-servo with time domain bounds

Input step disturbance    Reference step




The integral action in the controller is lost, just as in lecture 11!

# Summary

- There are efficient algorithms for convex optimization, e.g.
  - Linear programming (LP)
  - Quadratic programming (QP)
  - Second order cone programming (SOCP)
  - Semi-definite programming (SDP)
- The Youla parametrization allows us to use these algorithms for control synthesis
- Resulting controllers have high order. Order reduction will be studied in the next lecture.