Course Outline

L1-L5 Specifications, models and loop-shaping by hand
L6-L8 Limitations on achievable performance
L9-L11 Controller optimization: Analytic approach
L12-L14 Controller optimization: Numerical approach

Lecture 8: Multivariable and Decentralized Control

- Transfer functions for MIMO-systems
  - vehicles
  - power network
  - process control industry
- Limitations due to unstable multivariable zeros
- Decentralized/decoupled control by pairing of signals
- Short warning on integral action in parallel systems

See "Lecture notes" and [G&L, Ch. 1 and 8.1–8.3]

Typical Process Control System

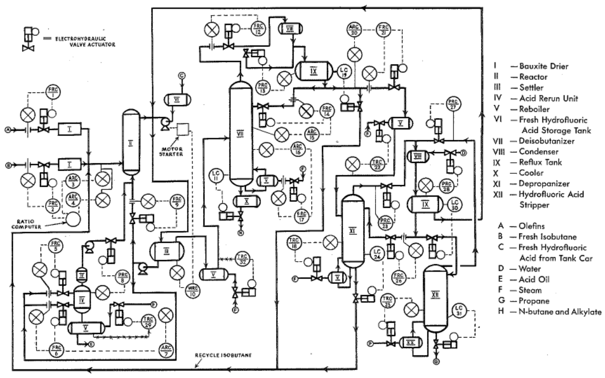


Figure 13–6. Automatic control system for Perco motor fuel alkylation process.

Example MIMO-system: A Distillation Column

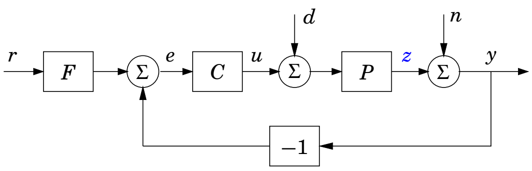
Example: Distillation column: raw oil inserted at bottom → different petro-chemical subcomponents extracted

$$\begin{bmatrix} Y_1(s) \\ Y_2(s) \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{4}{50s+1}e^{-27s} & \frac{1.8}{60s+1}e^{-28s} & \frac{5.9}{50s+1}e^{-27s} \\ \frac{5.4}{50s+1}e^{-18s} & \frac{5.7}{60s+1}e^{-14s} & \frac{6.9}{40s+1}e^{-15s} \end{bmatrix}}_{P(s)} \begin{bmatrix} U_1(s) \\ U_2(s) \\ U_3(s) \end{bmatrix}$$

Outputs:
$y_1 =$ top draw composition
$y_2 =$ side draw composition

Inputs:
$u_1 =$ top draw flowrate
$u_2 =$ side draw flowrate
$u_3 =$ bottom temperature control input

Multivariable transfer functions



Order matters!!

$$Z(s) = PCF \cdot R(s) + P \cdot D(s) - PC \cdot [N(s) + Z(s)]$$
$$[I + PC]Z(s) = PCF \cdot R(s) + P \cdot D(s) - PC \cdot N(s)$$

$$Z(s) = [I + PC]^{-1} \cdot (PCF \cdot R(s) + P \cdot D(s) - PC \cdot N(s))$$

Notice that $[I + PC]^{-1}$ is generally not the same as $[I + CP]^{-1}$.

Sensitivity functions for MIMO-systems

Output sensitivity function
$S = (I + PC)^{-1}$ $\qquad G_{?\to ?}$

Input sensitivity function
$(I + CP)^{-1}$ $\qquad G_{?\to ?}$

Complementary sensitivity function
$T = (I + PC)^{-1}PC$ $\qquad G_{?\to ?}$

1-minute problem:

Find the transfer functions above in the block diagram on the previous slide. (Extra: What are the dimensions?)

Some useful math relations

Notice the following identities:

(i) $[I + PC]^{-1}P = P[I + CP]^{-1}$

(ii) $C[I + PC]^{-1} = [I + CP]^{-1}C$

(iii) $T = P[I + CP]^{-1}C = PC[I + PC]^{-1} = [I + PC]^{-1}PC$

(iv) $S + T = I$

Proof:
The first equality follows by multiplication on both sides with $(I + PC)$ from the left and with $(I + CP)$ from the right.

Left: $[I + PC][I + PC]^{-1}P[I + CP] = I \cdot [P + PCP] = [I + PC]P$
Right: $[I + PC]P[I + CP]^{-1}[I + CP] = [I + PC]P \cdot I = [I + PC]P$

–"Push through and keep track of order"

Lecture 8: Multivariable and Decentralized Control

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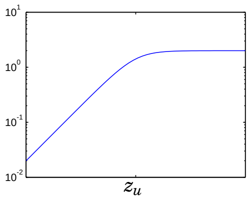
- Short warning on integral action in parallel systems

Hard limitations from unstable zeros

If the plant has an unstable zero z_u, then the specification

$$\left\| [I + P(i\omega)C(i\omega)]^{-1} \right\| < \frac{\sqrt{2}}{\sqrt{1 + z_u^2/\omega^2}} \qquad \text{for all } \omega$$

is impossible to satisfy.

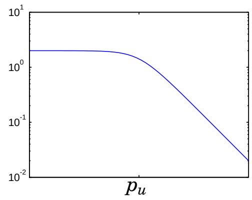


Hard limitations from unstable poles

If the plant has an unstable pole p_u, then the specification

$$\left\| P(i\omega)C(i\omega)[I + P(i\omega)C(i\omega)]^{-1} \right\| < \frac{\sqrt{2}p_u}{\sqrt{\omega^2 + p_u^2}} \qquad \text{for all } \omega$$

is impossible to satisfy.



Non-minimum phase MIMO System

Example [G&L, Ch 1]
Consider a feedback system $Y(s) = (I + PC)^{-1} \cdot R(s)$ with the multivariable process

$$P(s) = \begin{bmatrix} \frac{2}{s+1} & \frac{3}{s+2} \\ \frac{1}{s+1} & \frac{1}{s+1} \end{bmatrix}$$

Computing the determinant

$$\det P(s) = \frac{2}{(s+1)^2} - \frac{3}{(s+2)(s+1)} = \frac{-s+1}{(s+1)^2(s+2)}$$

shows that the process has an unstable zero at $s = 1$, which will limit the achievable performance.

See lecture notes for details of the following slides (checking three different controllers)

Example — controller 1

The controller

$$C_1(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & -\frac{3K_2(s+0.5)}{s(s+2)} \\ -\frac{K_1(s+1)}{s} & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

gives the diagonal loop transfer matrix

$$P(s)C_1(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & 0 \\ 0 & \frac{K_2(s+0.5)(-s+1)}{s(s+1)(s+2)} \end{bmatrix}$$

Hence the system is decoupled into to scalar loops, each with an unstable zero at $s = 1$ that limits the bandwidth.

The closed loop step responses are shown in Figure **??**.

Step responses using controller 1

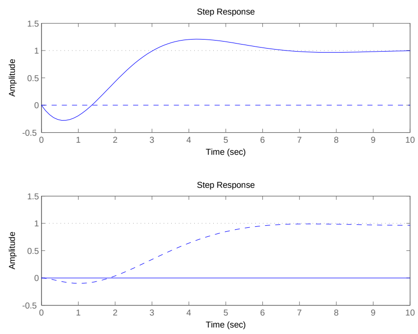


Figure : Closed loop step responses with decoupling controller $C_1(s)$ for the two outputs y_1 (solid) and y_2 (dashed). The upper plot is for a reference step for y_1. The lower plot is for a reference step for y_2.

Example – controller 2

The controller

$$C_2(s) = \begin{bmatrix} \frac{K_1(s+1)}{s} & K_2 \\ -\frac{K_1(s+1)}{s} & K_2 \end{bmatrix}$$

gives the diagonal loop transfer matrix

$$P(s)C_2(s) = \begin{bmatrix} \frac{K_1(-s+1)}{s(s+2)} & \frac{K_2(5s+7)}{(s+2)(s+1)} \\ 0 & \frac{2K_2}{s+1} \end{bmatrix}$$

Now the decoupling is only partial:
Output y_2 is not affected by r_1. Moreover, there is no unstable zero that limits the rate of response in y_2!

The closed loop step responses for $K_1 = 1$, $K_2 = 10$ are shown in Figure **??**.

Step responses using controller 2

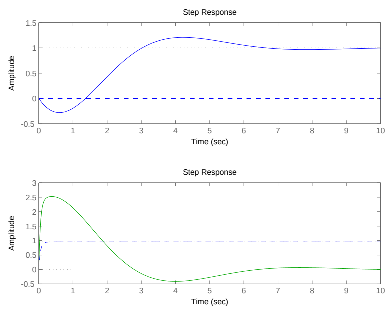


Figure : Closed loop step responses with controller $C_2(s)$ for the two outputs y_1 (solid) and y_2 (dashed). The right half plane zero does not prevent a fast y_2-response to r_2 but at the price of a simultaneous undesired response in y_1.

Example – controller 3

The controller

$$C_3(s) = \begin{bmatrix} K_1 & \frac{-K_2(s+0.5)}{s(s+2)} \\ K_1 & \frac{2K_2(s+0.5)}{s(s+1)} \end{bmatrix}$$

gives the diagonal loop transfer matrix

$$P(s)C_3(s) = \begin{bmatrix} \frac{K_1(5s+7)}{(s+1)(s+2)} & 0 \\ \frac{2K_1}{s+1} & \frac{K_2(-1+s)(s+0.5)}{s(s+1)^2(s+2)} \end{bmatrix}$$

In this case y_1 is decoupled from r_2 and can respond arbitrarily fast for high values of K_1, at the expense of bad behavior in y_2. Step responses for $K_1 = 10$, $K_2 = -1$ are shown in Figure **??**.

Step responses using controller 3

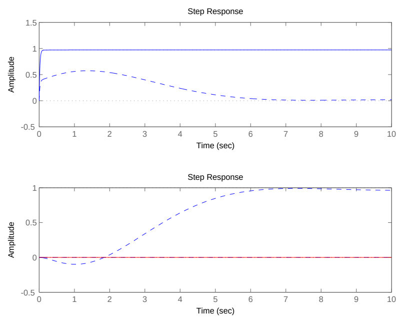


Figure : Closed loop step responses with controller $C_3(s)$ for the two outputs y_1 (solid) and y_2 (dashed). The right half plane zero does not prevent a fast y_1-response to r_1 but at the price of a simultaneous undesired response in y_2.

Example — summary

To summarize, the example shows that even though a **multivariable unstable zero always gives a performance limitation**, it is **possible to influence** where the effects should show up.

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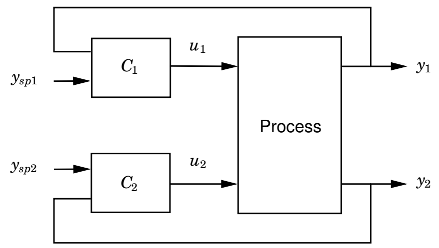
Interaction of Simple Loops



$$
\begin{aligned}
Y_1(s) &= p_{11}(s)U_1(s) + p_{12}U_2(s) \\
Y_2(s) &= p_{21}(s)U_1(s) + p_{22}U_2(s),
\end{aligned}
$$

What happens when the controllers are tuned individually?

Rosenbrock's Example

There is a nice collection of linear multivariable systems with interesting properties. Here is one of them

$$
P(s) = \begin{pmatrix} \dfrac{1}{s+1} & \dfrac{2}{s+3} \\ \dfrac{1}{s+1} & \dfrac{1}{s+1} \end{pmatrix}
$$

Very benign subsystems (compare with example in [G&L, Ch.1]).

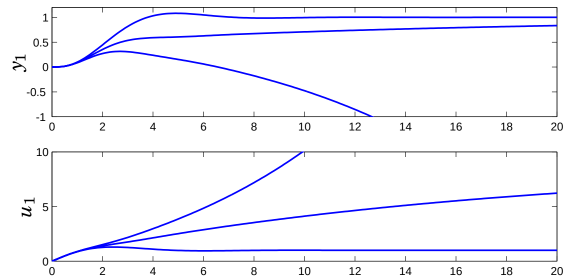
The transmission zeros are given by

$$
\det P(s) = \frac{1}{s+1}\left(\frac{1}{s+1} - \frac{2}{s+3}\right) = \frac{1-s}{(s+1)^2(s+3)} = 0.
$$

Difficult to control the system with gain crossover frequencies larger than $\omega_{gc} = 0.5$.

An Example

Controller C_1 is a PI controller with gains $k_1 = 1$, $k_i = 1$, and the C_2 is a proportional controller with gains $k_2 = 0$, 0.8, and 1.6.



The second controller has a major impact on the first loop!

RGA / Bristol's Relative Gain

- A simple way of measuring interaction based on static properties
- Edgar H. Bristol, "On a new measure of interaction for multivariable process control", [IEEE TAC 11(1967) pp. 133–135]
- Idea: What is effect of control of one loop on the steady state gain of another loop?
- Consider one loop when the other loop is under perfect control

$$
\begin{aligned}
Y_1(s) &= p_{11}(s)U_1(s) + p_{12}U_2(s) \\
0 &= p_{21}(s)U_1(s) + p_{22}U_2(s).
\end{aligned}
$$

RGA / Bristol's Relative Gain

Consider the first loop $u_1 \to y_1$ when the second loop is in perfect control ($y_2 = 0$)

$$
\begin{aligned}
Y_1(s) &= p_{11}(s)U_1(s) + p_{12}U_2(s) \\
0 &= p_{21}(s)U_1(s) + p_{22}U_2(s).
\end{aligned}
$$

Eliminating $U_2(s)$ from the first equation gives

$$
Y_1(s) = \frac{p_{11}(s)p_{22}(s) - p_{12}(s)p_{21}(s)}{p_{22}(s)}U_1(s).
$$

The ratio of the static gains of loop 1 when the second loop is open and closed is

$$
\lambda = \frac{p_{11}(0)p_{22}(0)}{p_{11}(0)p_{22}(0) - p_{12}(0)p_{21}(0)}.
$$

Parameter λ is called Bristol's interaction index

Bristol's Relative Gain Array (RGA)

Let $P(s)$ be an $n \times n$ matrix of transfer functions. The relative gain array is

$$\Lambda = P(0). \star P^{-T}(0)$$

The product $.\star$ is "element-by-element product" (Schur or Hadamard product, same notation in matlab). Properties

- $(A. \star B)^T = A^T. \star B^T$
- P diagonal or triangular gives $\Lambda = I$
- Not effected by diagonal scalings

Insight and use

- A measure of static interactions for square systems which tells how the gain in one loop is influenced by perfect feedback on all other loops
- Dimension free. Row and column sums are 1.
- Negative elements correspond to sign reversals due to feedback of other loops

RGA in Control

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = P \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} \qquad \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix} = P^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}$$

- P_{kj} gives the map $u_j \to y_k$ when $u_i = 0$ for $i \neq j$
- $[P^{-1}]_{jk}$ gives the map $y_k \to u_j$ when $y_i = 0$ for $i \neq k$

If $[RGA(P)]_{k,j} = 1$, then only y_k is needed to recover u_j. This means strong coupling and u_j is a natural input for control of y_k.

Pairing

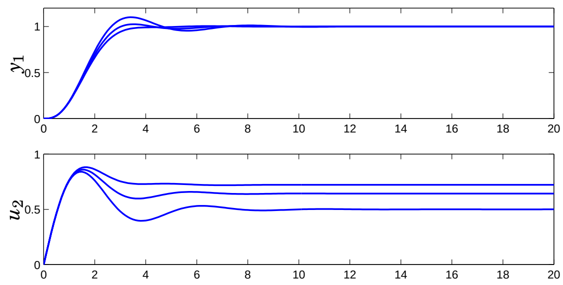
When designing complex systems loop by loop we must decide what measurements should be used as inputs for each controller. This is called the pairing problem. The choice can be governed by physics but the relative gain can also be used

Consider the previous example

$$P(0) = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \qquad P^{-1}(0) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$
$$\Lambda = P(0). \star P^{-T}(0) = \begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix},$$

- Negative sign indicates the sign reversal found previously
- Better to use reverse pairing, i.e. let u_2 control y_1

Step Responses with Reverse Pairing



- $U_2 = \left(1 + \frac{1}{s}\right)(Y_{\text{ref1}} - Y_1)$
- $u_1 = -k_2 y_2$ with $k_2 = 0$, 0.8, and 1.6.

Summary for 2×2 Systems (RGA)

$\lambda = 1$ No interaction

$\lambda = 0$ Closed loop gain $u_1 \to y_1$ is zero. Avoid this.

$0 < \lambda < 1$ Closed loop gain $u_1 \to y_1$ is larger than open loop gain.

$\lambda > 1$ Closed loop gain $u_1 \to y_1$ is smaller than open loop gain. Interaction increases with increasing λ. Very difficult to control both loops independently if λ is very large.

$\lambda < 0$ The closed loop gain $u_1 \to y_1$ has different sign than the open loop gain. Opening or closing the second loop has dramatic effects. The loops are counteracting each other. Such pairings should be avoided for decentralized control and the loops should be controlled jointly as a multivariable system.

Interactions Can be Beneficial

$$P(s) = \begin{pmatrix} p_{11}(s) & p_{12}(s) \\ p_{21}(s) & p_{22}(s) \end{pmatrix} = \begin{pmatrix} \frac{s-1}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \\ \frac{-6}{(s+1)(s+2)} & \frac{s-2}{(s+1)(s+2)} \end{pmatrix}.$$

The relative gain array

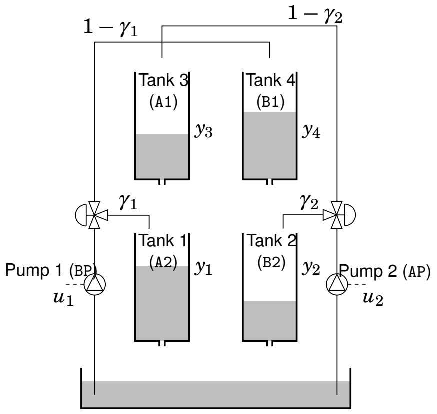
$$R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

Transmission zeros

$$\det P(s) = \frac{(s-1)(s-2)+6s}{(s+1)^2(s+2)^2} = \frac{s^2+4s+2}{(s+1)^2(s+2)^2}$$

Difficult to control individual loops fast because of the zero at $s = 1$. Since there are no multivariable zeros in the RHP the multivariable system can easily be controlled fast but this system is not robust to loop breaks.

The Quadruple Tank



Transfer Function of Linearized Model

Transfer function from u_1, u_2 to y_1, y_2

$$P(s) = \begin{pmatrix} \frac{\gamma_1 c_1}{1+sT_1} & \frac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)} \\ \frac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)} & \frac{\gamma_2 c_2}{1+sT_2} \end{pmatrix}$$

Transmission zeros

$$\det P(s) = \frac{(1+sT_3)(1+sT_4) - \frac{(1-\gamma_1)(1-\gamma_2)}{\gamma_1\gamma_2}}{(1+sT_1)(1+sT_2)(1+sT_3)(1+sT_4)}$$

- No interaction of $\gamma_1 = \gamma_2 = 1$
- Minimum phase if $1 \leq \gamma_1 + \gamma_2 \leq 2$
- Nonminimum phase if $0 < \gamma_1 + \gamma_2 \leq 1$.
- Intuition?

Relative Gain Array

Zero frequency gain matrix

$$P(0) = \begin{pmatrix} \gamma_1 c_1 & (1-\gamma_2)c_1 \\ (1-\gamma_1)c_2 & \gamma_2 c_2 \end{pmatrix}$$

The relative gain array

$$RGA(P(0)) = \begin{pmatrix} \lambda & 1-\lambda \\ 1-\lambda & \lambda \end{pmatrix}$$

where

$$\lambda = \frac{\gamma_1 \gamma_2}{\gamma_1 + \gamma_2 - 1}$$

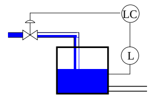
- No interaction for $\gamma_1 = \gamma_2 = 1$
- Severe interaction if $\gamma_1 + \gamma_2 < 1$

A multivariable control problem

-The water is too cold! -Now it is too hot! -Now it is too cold! -Now it is too deep!

How to do if we want to separate control of

- temperature?
- water level?



Decoupling

Simple idea: Find a compensator so that the system appears to be without coupling ("block-diagonal transfer function matrix").

Many versions – here we will consider

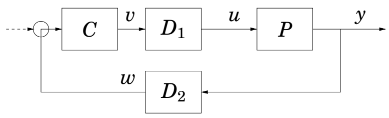
- Input decoupling $Q = PD_1$
- Output decoupling $Q = D_2 P$
- "both" $Q = D_2 P D_1$

but many different methods including

- Conventional (Feedforward)
- Inverse (Feedback)
- Static

Important to consider windup, manual control and mode switches.

- Keep the decentralized philosophy



Find D_1 and D_2 so that the controller sees a "diagonal plant":

$$D_2 P D_1 = \begin{bmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{bmatrix}$$

Then we can use a "decentralized" controller C with same block-diagonal structure.

Decoupling —- Flight Control

- Longitudinal
- Lateral

May be good to decouple interaction to outputs, but you should also be careful not to waste control action to "strange decouplings"!!

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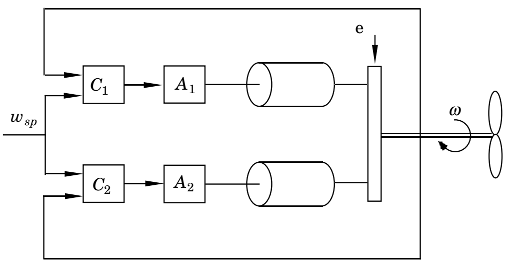
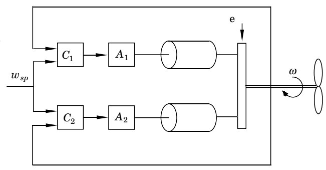
Systems with Parallel Actuation



- Motor drives for papermachines and rolling mills
- Trains with several motors or several coupled trains
- Power systems

A Prototype Example

$$J\frac{d\omega}{dt} + D\omega = M_1 + M_2 - M_L,$$



Proportional control

$$M_1 = M_{10} + K_1(\omega_{sp} - \omega)$$
$$M_2 = M_{20} + K_2(\omega_{sp} - \omega)$$

The proportional gains tell how the load is distributed

$$J\frac{d\omega}{dt} + (D + K_1 + K_2)\omega = M_{10} + M_{20} - M_L + (K_1 + K_2)\omega_{sp}.$$

A first order system with time constant $T = J/(D + K_1 + K_2)$

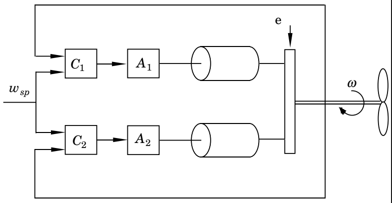
Discuss response speed, damping and steady state

$$\omega = \omega_0 = \frac{K_1 + K_2}{D + K_1 + K_2}\omega_{sp} + \frac{M_{10} + M_{20} - M_L}{D + K_1 + K_2}.$$

Integral Action?

What if we instead use two PI-controllers?

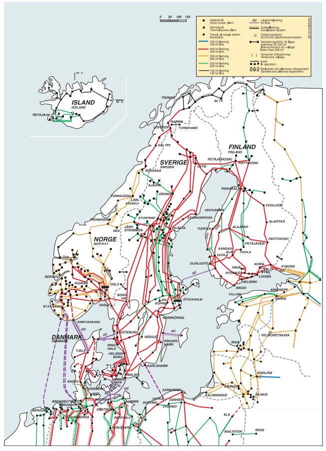
WARNING!!!



Prototypes for lack of controllability and observability!

Power Systems - Massive Parallellism

- Edison's experience
  - Two generators with governors having integral action
- Many generators supply power to the net.
  - Frequency control
  - Voltage control
- Isochronous governors (integral action) and governors with speed-drop (no integral action)



Summary

- All real systems are coupled
- Multivariable zeros - limitations
  - Never forget process redesign
- Relative gain array and singular values give insight
- Why decouple
  - Simple system.
  - SISO design, tuning and operation can be used
  - What is lost?
- Parallel systems
  - One integrator only!
- Next lecture: Multivariable design LQ/LQG